

Good Reductions of Shimura Varieties of Preabelian Type in Arbitrary Unramified Mixed Characteristic, I

Adrian Vasiu, UA, 11/4/03

ABSTRACT. We prove the existence of weak integral canonical models of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic $(0, p)$. As a first application we solve a conjecture of Langlands for Shimura varieties of Hodge type. As a second application we prove the existence of integral canonical models of Shimura varieties of preabelian (resp. of abelian) type in mixed characteristic $(0, p)$ with $p \geq 3$ (resp. with $p = 2$) and with respect to hyperspecial subgroups; if $p = 3$ (resp. if $p = 2$) we restrict in this part I either to the $A_n, C_n, D_n^{\mathbb{H}}$ (resp. A_n and C_n) types or to the B_n and $D_n^{\mathbb{R}}$ (resp. $B_n, D_n^{\mathbb{H}}$ and $D_n^{\mathbb{R}}$) types which have compact factors over \mathbb{R} (resp. which have compact factors over \mathbb{R} in some p -compact sense). Though the second application is new just for $p \leq 3$, a great part of its proof is new even for $p \geq 5$ and corrects [Va1, 6.4.11] in most of the cases. The second application forms progress towards the proof of a conjecture of Milne. It also provides in arbitrary mixed characteristic the very first examples of general nature of projective varieties over number fields which are not embeddable into abelian varieties and which have Néron models over certain local rings of integers of number fields.

Key words: Shimura varieties, reductive group schemes, abelian schemes, integral models, p -divisible groups and F -crystals.

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§1. Introduction

For a module M over a commutative ring with unit R , let $M^* := \text{Hom}_R(M, R)$ and let $GL(M)$ be the group scheme over R of linear automorphisms of M . A bilinear form on M is called perfect if it induces an isomorphism $M \xrightarrow{\sim} M^*$. If M is a free R -module of even rank and if ψ_M is a perfect alternating form on M , then $Sp(M, \psi_M)$ and $GSp(M, \psi_M)$ are viewed as reductive group schemes. If $*_R$ or $*$ is an object or a morphism of the category of R -schemes, let $*_U$ be its pull back via an affine morphism $m_R : \text{Spec}(U) \rightarrow \text{Spec}(R)$. If

there are several ways to take m_R , we mention the homomorphism $R \rightarrow U$ defining m_R but often we do not add it as part of the notation $*_U$.

If F is a reductive group scheme over a scheme Y , then its fibres are connected and $Z(F)$, F^{der} , F^{ad} and F^{ab} denote the center, the derived group, the adjoint group and respectively the abelianization of F . So $Z^{\text{ab}}(F) = F/F^{\text{der}}$ and $F^{\text{ad}} = F/Z(F)$. Let $Z^0(F)$ be the maximal subtorus of $Z(F)$. If $Y = \text{Spec}(R)$ is affine, let $\text{Lie}(F)$ be the R -Lie algebra of F and let \mathbb{G}_{mR} be the rank 1 split torus over R . Similarly, the group schemes \mathbb{G}_{aR} , SL_{nR} , etc., will be understood to be over R . If $F_1 \hookrightarrow F$ is a monomorphism of group schemes over R which is a closed embedding, then we identify F_1 with its image in F and we consider intersections of subgroups of $F_1(R)$ with subgroups of $F(R)$.

If $Y \rightarrow Z$ is an étale cover, then $\text{Res}_{Y/Z}F$ is the reductive group scheme over Z obtained from F through the Weil restriction of scalars (see [BLR, 7.6] and [BrT, 1.5]). Let $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m\mathbb{C}}$. We identify $\mathbb{S}(\mathbb{R}) = \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C}) = \mathbb{G}_{m\mathbb{C}}(\mathbb{C}) \times \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ in such a way that the monomorphism $\mathbb{R} \hookrightarrow \mathbb{C}$ induces the map $z \rightarrow (z, \bar{z})$, where $z \in \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$.

Let p be a rational prime. Let $\mathbb{Z}_{(p)}$ be the localization of \mathbb{Z} with respect to its ideal (p) . If E is a number field, let $E_{(p)}$ be the normalization of $\mathbb{Z}_{(p)}$ in E . For v a finite prime of E , let E_v be the completion of E with respect to v and let $e(v)$ be the index of ramification of v . Let $W(k)$ be the Witt ring of a perfect field k of characteristic p . Always $n \in \mathbb{N}$. Let $\mathbb{A}_f := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ be the topological ring of finite adèles of \mathbb{Q} . Warning: let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of \mathbb{Q} with the p -component omitted; so $\mathbb{A}_f = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$.

If $R \in \{\mathbb{A}_f, \mathbb{A}_f^{(p)}, \mathbb{Q}_p\}$, then the group $F(R)$ is endowed with the coarsest topology making all maps $R = \mathbb{G}_{aR}(R) \rightarrow F(R)$ associated to morphisms $\mathbb{G}_{aR} \rightarrow F$ of R -schemes, to be continuous; so $F(R)$ is a totally discontinuous locally compact group.

A continuous action of a totally discontinuous locally compact group on a scheme is always in the sense of [De2, 2.7.1] and is a right action.

1.1. A review of Shimura varieties. A Shimura pair (G, X) comprises from a reductive group G over \mathbb{Q} and a $G(\mathbb{R})$ -conjugacy class X of homomorphisms $\mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying Deligne's axioms of [De2, 2.1]: the Hodge \mathbb{Q} -structure on $\text{Lie}(G)$ defined by any $h \in X$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$, $\text{Ad} \circ h(i)$ is a Cartan involution of $\text{Lie}(G_{\mathbb{R}}^{\text{ad}})$, and no simple factor of G^{ad} becomes compact over \mathbb{R} . Here $\text{Ad} : G_{\mathbb{R}} \rightarrow GL(\text{Lie}(G_{\mathbb{R}}^{\text{ad}}))$ is the adjoint representation. These axioms imply that X has a natural structure of a hermitian symmetric domain, cf. [De2, 1.1.17]. For $h \in X$ we consider the Hodge cocharacter

$$\mu_h : \mathbb{G}_{m\mathbb{C}} \rightarrow G_{\mathbb{C}}$$

defined on complex points by the rule: $z \in \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ is mapped into $h_{\mathbb{C}}(z, 1)$.

For generalities on Shimura pairs and varieties and their types we refer to [De2], [Mi2, §1], [Mi3] and [Va1, 2.2 to 2.8]. The most studied Shimura pairs are constructed as follows. Let ψ be a non-degenerate symplectic form on an even dimensional vector space W over \mathbb{Q} . Let S be the set of all monomorphisms $\mathbb{S} \hookrightarrow GSp(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$ defining Hodge \mathbb{Q} -structures on W of type $\{(-1, 0), (0, -1)\}$ and having either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. The pair $(GSp(W, \psi), S)$ is a Shimura pair defining a Siegel modular variety. Let L be a \mathbb{Z} -lattice of W such that ψ induces a perfect form $\psi : L \otimes_{\mathbb{Z}} L \rightarrow \mathbb{Z}$. Let $K_p := GSp(L, \psi)(\mathbb{Z}_p)$.

The adjoint (resp. the toric part) of a Shimura pair (G, X) is denoted as $(G^{\text{ad}}, X^{\text{ad}})$ (resp. $(G^{\text{ab}}, X^{\text{ab}})$), cf. [Va1, 2.4.1]. We say that (G, X) is of Hodge type if there is an injective map $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$ for some symplectic space (W, ψ) over \mathbb{Q} as above. So $f : G \hookrightarrow GSp(W, \psi)$ is a monomorphism such that $f_{\mathbb{R}} \circ h \in S$, $\forall h \in X$.

We say that (G, X) is of preabelian type if its adjoint $(G^{\text{ad}}, X^{\text{ad}})$ is isomorphic to the adjoint of a Shimura pair (G_1, X_1) of Hodge type. If moreover, we can choose (G_1, X_1) such that G_1^{der} is an isogeny cover of G^{der} , then we say that (G, X) is of abelian type.

The adjoint Shimura pair $(G^{\text{ad}}, X^{\text{ad}})$ is called simple if G^{ad} is a simple \mathbb{Q} -group. In this case all simple factors of $G_{\mathbb{C}}^{\text{ad}}$ have the same Lie type LT . If LT is not D_n for some $n \geq 4$, then $(G^{\text{ad}}, X^{\text{ad}})$ is said to be of LT type. Let now LT be D_n with $n \geq 4$. If $n \geq 5$, then $(G^{\text{ad}}, X^{\text{ad}})$ is said to be of $D_n^{\mathbb{H}}$ (resp. $D_n^{\mathbb{R}}$) type iff all simple, non-compact factors of $G_{\mathbb{R}}^{\text{ad}}$ are isomorphic to $SO^*(2n)_{\mathbb{R}}^{\text{ad}}$ (resp. $SO(2, 2n-2)_{\mathbb{R}}^{\text{ad}}$); see [He, p. 445] for these classical groups. If $n \geq 5$ and if $(G^{\text{ad}}, X^{\text{ad}})$ is not of $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type, then $(G^{\text{ad}}, X^{\text{ad}})$ is said to be of D_n^{mixed} mixed type. See [De2, p. 272] or [Va4, 2.2.1] for the difference between the D_4^{mixed} , $D_4^{\mathbb{H}}$ and $D_4^{\mathbb{R}}$ types. If $(G^{\text{ad}}, X^{\text{ad}})$ is also of abelian type, then a classical result of Satake says that LT is a classical Lie type (see [Sa]). Deligne proved that a simple, adjoint Shimura pair is of abelian type iff it is of A_n , B_n , C_n , $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type (cf. [De2, 2.3.10]).

Let $E(G, X) \hookrightarrow \mathbb{C}$ be the number subfield of \mathbb{C} which is the field of definition of the $G(\mathbb{C})$ -conjugacy class of the cocharacters μ_h 's of $G_{\mathbb{C}}$, cf. [Mi2, p. 163–164]. We recall that $E(G, X)$ is called the reflex field of (G, X) . Let $O_{(v)}$ be the localization of the ring of integers $O(G, X)$ of $E(G, X)$ with respect to a prime v of $E(G, X)$ dividing p . Let $k(v)$ be the residue field of v .

The Shimura variety $\text{Sh}(G, X)$ is identified with the canonical model over $E(G, X)$ of the complex Shimura variety

$$\text{Sh}(G, X)_{\mathbb{C}} := \text{proj.lim.}_{K \in CO(G)} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K,$$

where $CO(G)$ is the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation (see [De1], [De2] and [Mi1] to [Mi4]). So $\text{Sh}(G, X)$ is an $E(G, X)$ -scheme together with a continuous action of $G(\mathbb{A}_f)$. For K a compact subgroup of $G(\mathbb{A}_f)$ let

$$\text{Sh}_K(G, X) := \text{Sh}(G, X) / K.$$

If $K \in CO(G)$, then a classical result of Baily and Borel allows us to view $\text{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$ as a complex variety and not just as a complex space (see [BB]).

Let H be a compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. We refer to the quadruple (G, X, H, v) as a Shimura quadruple. The group $G_{\mathbb{Q}_p}$ is called unramified iff it has a Borel subgroup and splits over an unramified, finite field extension of \mathbb{Q}_p . See [Ti2] or 2.2 for hyperspecial subgroups. Here we will add just two things. First $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ has hyperspecial subgroups iff $G_{\mathbb{Q}_p}$ is unramified. Second H is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ iff it is the group of \mathbb{Z}_p -valued points of a reductive group scheme over \mathbb{Z}_p having $G_{\mathbb{Q}_p}$ as its generic fibre.

In this paper we study integral models of $\text{Sh}_H(G, X)$ over different localizations of $E_{(p)}$ (like over $O_{(v)}$). The subject has a long history, the first fundamental result being the construction of (coarse) moduli schemes over \mathbb{Z} of principally polarized abelian schemes endowed with level structures obtained by Mumford in 1965 (see [MFK, Ths. 7.9 and 7.10]).

In 1976 Langlands conjectured the existence of a good integral model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$, provided H is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (see [La, p. 411]). But only in 1992, an idea of Milne made it significantly clearer how to characterize and identify the “good” integral models. We now recall some definitions from [Va1] which are just slight modifications of the original ideas of Milne expressed in definitions [Mi2, 2.1, 2.2, 2.5 and 2.9].

1.1.1. Definitions. Let $*$ be a faithfully flat $\mathbb{Z}_{(p)}$ -algebra which is a localization of $E_{(p)}$.

1) A smooth (resp. normal) integral model of $\mathrm{Sh}_H(G, X)$ over $*$ is a faithfully flat $*$ -scheme \mathcal{N} together with a continuous action of $G(\mathbb{A}_f^{(p)})$ on it such that the following two properties hold:

(i) *its generic fibre $\mathcal{N}_{E(G, X)}$ with its induced $G(\mathbb{A}_f^{(p)})$ -action is $\mathrm{Sh}_H(G, X)$ with its canonical $G(\mathbb{A}_f^{(p)})$ -action;*

(ii) *there is a compact, open subgroup H_0 of $G(\mathbb{A}_f^{(p)})$ with the property that for any inclusion $H_1 \leq H_2$ of open subgroups of H_0 , the canonical morphism $\mathcal{N}/H_1 \rightarrow \mathcal{N}/H_2$ induced by the action of $G(\mathbb{A}_f^{(p)})$ on \mathcal{N} , is an étale morphism between smooth (resp. normal), separated schemes of finite type over $*$.*

The faithfully flat $*$ -scheme \mathcal{N} is separated (cf. (ii)) and so from (i) we get that the action of $G(\mathbb{A}_f^{(p)})$ on \mathcal{N} is uniquely determined; so often we will not mention explicitly this action and we will refer just to \mathcal{N} itself as a smooth (resp. normal) integral model of $\mathrm{Sh}_H(G, X)$ over $*$. Also if one (any) such H_0 is such that \mathcal{N}/H_0 is a smooth (resp. normal), (quasi-) projective $*$ -scheme, then we say \mathcal{N} is a pro-étale cover of a smooth (resp. normal), (quasi-) projective $*$ -scheme.

2) Let Y be a faithfully flat $\mathbb{Z}_{(p)}$ -scheme which is regular. We say Y is healthy regular if for any open subscheme U of Y containing $Y_{\mathbb{Q}}$ and whose complement in Y is of codimension at least 2, every abelian scheme over U extends to an abelian scheme over Y . A flat $*$ -scheme Z is said to have the extension property if for any healthy regular scheme \tilde{Y} over $*$, every morphism $\tilde{Y}_{E(G, X)} \rightarrow Z_{E(G, X)}$ extends uniquely to a morphism $\tilde{Y} \rightarrow Z$.

3) We say that a smooth integral model \mathcal{N} of $\mathrm{Sh}_H(G, X)$ over $*$ is an integral canonical model of $\mathrm{Sh}_H(G, X)$ over $*$ if \mathcal{N} (viewed just as a scheme) has the extension property and is a healthy regular scheme.

For practical reasons we add the following definitions.

1.2. Definitions. 1) A smooth integral model \mathcal{N} of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$ is called a weak integral canonical model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$ if it has the following smooth extension property: for any regular, formally smooth scheme Y over a DVR O faithfully flat over $O_{(v)}$ and of the same index of ramification as $O_{(v)}$, every morphism $Y_{E(G, X)} \rightarrow \mathcal{N}_{E(G, X)} = \mathrm{Sh}_H(G, X)$ of $E(G, X)$ -schemes extends uniquely to a morphism $Y \rightarrow \mathcal{N}$ of $O_{(v)}$ -schemes.

2) We say (G, X) has compact factors if the group $G_{\mathbb{R}}^{\mathrm{ad}}$ is non-trivial and has simple factors which are compact.

3) We assume that $e(v) = 1$. An affine, flat group scheme $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ having G as its generic fibre is called a quasi-reductive group scheme for (G, X, v) if there is a reductive, normal subgroup $G_{\mathbb{Z}_p}^1$ of $G_{\mathbb{Z}_p}$ such that there is a cocharacter $\mu_v : \mathbb{G}_{mW(k(v))} \rightarrow G_{W(k(v))}^1$ whose extension to \mathbb{C} via an (any) $O_{(v)}$ -monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ is $G(\mathbb{C})$ -conjugate with the cocharacters μ_h ($h \in X$) defined in the beginning of 1.1.

4) We say the prime v is p -compact for (G, X) if either G^{ad} is trivial or the product decomposition $G_{\mathbb{Q}_p}^{\text{ad}} = \prod_{i \in I_p} G_i$ in simple, adjoint \mathbb{Q}_p -groups is such that $\forall i \in I_p$, the pull back of G_i to $\text{Spec}(\mathbb{C})$ via an (any) $E(G, X)$ -embedding $E(G, X)_v \hookrightarrow \mathbb{C}$ has simple factors which are pull backs of simple, compact factors of $G_{\mathbb{R}}^{\text{ad}}$.

It is easy to see that any (weak) integral canonical model is uniquely determined up to unique isomorphism. Moreover, if $e(v) \leq \max\{1, p-2\}$, then any integral canonical model of $\text{Sh}_H(G, X)$ over $O_{(v)}$ is also a weak integral canonical model. This is so due to the following theorem (see [Va1, 3.2.2 1]) for $p > 2$ and see [Va2, 1.3] for the case when $e(v) = 1$).

1.2.1. Theorem. *If $e(v) \leq \max\{1, p-2\}$, then any regular, formally smooth scheme over a DVR which is a faithfully flat $O_{(v)}$ -algebra and which is of index of ramification at most $\max\{1, p-2\}$, is healthy regular.*

From 1.2.1 and [Va1, rm. 3.2.3.1 2)] we get that for $e(v) \leq \max\{1, p-2\}$ and for $* = O_{(v)}$, the definition 1.1.1 3) coincides with the definition [Va1, 3.2.3 6)]. So as in this paper we deal in essence just with the case $e(v) = 1$ and with normal models which are separated, we will not make in what follows any distinction between the two possible definitions of integral canonical models.

1.3. Basic notations. Let $(GSp(W, \psi), S)$, L and K_p be as in 1.1. Let $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. For the rest of this introduction we will assume that we have an injective map

$$f : (G, X) \hookrightarrow (GSp(W, \psi), S)$$

and that $H := K_p \cap G(\mathbb{Q}_p)$. Let $G_{\mathbb{Z}_{(p)}}$ be the Zariski closure of G in $GL(L_{(p)})$. So $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Let $E(G, X)$, v , $k(v)$ and $O_{(v)}$ be as in 1.1. Let

$$\mathcal{M}$$

be the integral canonical model of $\text{Sh}_{K_p}(GSp(W, \psi), S)$ over $\mathbb{Z}_{(p)}$. It is known that \mathcal{M} is the moduli scheme over $\mathbb{Z}_{(p)}$ parameterizing isomorphism classes of principally polarized abelian schemes of relative dimension $\dim_{\mathbb{Q}}(W)/2$ and having level N symplectic similitude structure, $\forall N \in \mathbb{N}$ prime to p (cf. [Va1, 3.2.9 and 4.1] and [MFK, Ths. 7.9 and 7.10]). See [Va1, 3.2.9 and 4.1] and [De1, 4.21] for the natural action of $GSp(W, \psi)(\mathbb{A}_f^{(p)})$ on \mathcal{M} . It is known that $\text{Sh}_H(G, X)$ is a finite scheme over $\mathcal{M}_{E(G, X)} = \text{Sh}_{K_p}(GSp(W, \psi), S)_{E(G, X)}$ (for instance, cf. 3.1.1 3)). If $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then in fact $\text{Sh}_H(G, X)$ is a closed subscheme of $\mathcal{M}_{E(G, X)}$ (cf. [Va1, 3.2.14]).

Let

$$\mathcal{N}'$$

be the normalization of $\mathcal{M}_{O(v)}$ in the ring of fractions of $\mathrm{Sh}_H(G, X)$; so if $G_{\mathbb{Z}(p)}$ is a reductive group scheme, then \mathcal{N}' is the normalization of the Zariski closure of $\mathrm{Sh}_H(G, X)$ in $\mathcal{M}_{O(v)}$. As $G(\mathbb{A}_f^{(p)})$ acts on $\mathrm{Sh}_H(G, X)$ and \mathcal{M} , we get a natural action of $G(\mathbb{A}_f^{(p)})$ on \mathcal{N}' . We consider the formally smooth locus

$$\mathcal{N}$$

of \mathcal{N}' over $O(v)$. It is a $G(\mathbb{A}_f^{(p)})$ -invariant, open subscheme of \mathcal{N}' such that $\mathcal{N}_{E(G, X)} = \mathcal{N}'_{E(G, X)}$, cf. 3.1.2. Let

$$(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$$

be the pull back to \mathcal{N}' of the universal abelian scheme over \mathcal{M} .

The main goal of this paper is to prove the following three basic Theorems.

1.4. Basic Theorem. *We assume that $e(v) = 1$ (i.e. that v is unramified over p) and that the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty. We have:*

1) *The $O(v)$ -scheme \mathcal{N} together with the natural action of $G(\mathbb{A}_f^{(p)})$ on it is a weak integral canonical model of $\mathrm{Sh}_H(G, X)$ over $O(v)$.*

2) *For any algebraically closed field k of characteristic p , the natural morphism $\mathcal{N}_{W(k)} \rightarrow \mathcal{M}_{W(k)}$ is a formally closed embedding at every k -valued point of $\mathcal{N}_{W(k)}$.*

1.5. Basic Theorem. *We assume that $e(v) = 1$ and that $G_{\mathbb{Z}(p)}$ is a quasi-reductive group scheme for (G, X, v) . If $p = 2$ we also assume that any pull back of \mathcal{A}' via a geometric point of $\mathcal{N}_{k(v)}$ has p -rank 0. Then $\mathcal{N}_{k(v)}$ is a non-empty, open closed subscheme of $\mathcal{N}'_{k(v)}$.*

1.6. Main Theorem. 1) *Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = \prod_{i \in I_0} (G_1^i, X_1^i)$ be the product decomposition in simple, adjoint Shimura pairs. We assume that $G_{\mathbb{Z}(p)}$ is a reductive group scheme and that one of the following three conditions holds:*

(i) $p \geq 5$;

(ii) $p = 3$ and $\forall i \in I_0$, the simple, adjoint Shimura pair (G_1^i, X_1^i) either is of A_n , C_n or $D_n^{\mathbb{H}}$ type or is of B_n or $D_n^{\mathbb{R}}$ type and moreover has compact factors;

(iii) $p = 2$ and $\forall i \in I_0$, the simple, adjoint Shimura pair (G_1^i, X_1^i) either is of A_n or C_n type or is of B_n , $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type and moreover the prime v_1^i of $E(G_1^i, X_1^i)$ divided by v is p -compact for (G_1^i, X_1^i) .

Then $e(v) = 1$ and $\mathcal{N} = \mathcal{N}'$ is an integral canonical model of $\mathrm{Sh}_H(G, X)$ over $O(v)$.

2) *Let (G_1, X_1) be a Shimura pair of preabelian type. Let p be a prime such that $G_{1\mathbb{Q}_p}$ is unramified; so $E(G_1, X_1)$ is unramified over p (cf. [Mi3, Cor. 4.7 (a)]). Let H_1 be a hyperspecial subgroup of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = \prod_{i \in I_0} (G_1^i, X_1^i)$ be the product decomposition in simple, adjoint Shimura pairs. We consider the set $S(G_1, X_1, 2)$ of primes v_1 of $E(G_1, X_1)$ dividing 2 and such that $\forall i \in I_0$ with the property that (G_1^i, X_1^i) is of B_n , $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type, the prime v_1^i of $E(G_1^i, X_1^i)$ divided by v is 2-compact for (G_1^i, X_1^i) . Let $O(G_1, X_1, 2)$ be the localization of $E(G_1, X_1)_{(2)}$ whose finite primes are precisely the primes in $S(G_1, X_1, 2)$. If $p \geq 3$ let $O(G_1, X_1, p) := E(G_1, X_1)_{(p)}$.*

If $p = 3$ we assume that condition (ii) of 1) holds. If $p = 2$ we assume that (G_1, X_1) is of abelian type and that the set $S(G_1, X_1, 2)$ is non-empty. Then there is an integral canonical model \mathcal{N}_1 of $\mathrm{Sh}_{H_1}(G_1, X_1)$ over $O(G_1, X_1, p)$ which is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme. If moreover (G_1^i, X_1^i) has compact factors for all $i \in I_0$, then \mathcal{N}_1 is a pro-étale cover of a smooth, projective $O(G_1, X_1, p)$ -scheme.

1.7. On contents. We detail on the contents of the paper. In 2.1 we list conventions and notations. In 2.2 we include some results on extending reductive group schemes; the main new result (see 2.2.3) is a purity type of theorem for the classical Lie types. In 2.3 we review properties of dilatations of affine group schemes over Witt rings. In §3 we list some properties of embeddings between Shimura pairs in a $\mathbb{Z}_{(p)}$ context.

In §4 we list different crystalline applications. In 4.1 and 4.2 we introduce basic notations and review two recent results pertaining to p -divisible groups and which play a central role in the rest of the paper. The results are: the extension theorem of de Jong (see [dJ2]) and a conjecture of Milne proved in [Va6, 1.2]. In 4.3 we prove 1.4. In 4.4 we present a simple criterion on when the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty. In 4.5 we apply 1.4 1) and 4.4 to get a solution of the mentioned conjecture of Langlands for the case of Shimura varieties of Hodge type. In 4.6 we refine the deformation theories of [Fa, §7] and [Va1, 5.4] as allowed by the recent tools developed in [Va6].

In §5 we use 2.2 and 4.1 to 4.3 to prove 1.5. The proof of 1.6 is carried on in 6.1 and 6.2. In 6.3 we use the projectiveness part of 1.6 2) to provide new examples of general nature of Néron models in the sense of [BLR, p. 12]. In 6.4 we include some complements; in particular in 6.4 5) and 6) we explain why we can view 1.5 as a generalization of [Mo] for the unramified context with $p > 2$.

1.8. More on literature. Referring to 1.4 1), in [No] it is shown that all ordinary points of $\mathcal{N}'_{k(v)}$ belong to $\mathcal{N}_{k(v)}$. Except the part involving projectiveness, Theorem 1.6 was proved for $p \geq 5$ in [Va1, 3.2.12, 5.1 and 6.4.1] (see also [Va5, 7.9.3] for the passage from the abelian type case to the preabelian type case) and for unitary Shimura varieties in [Va4, §5]. The works [MFK], [Dr], [Mo], [Zi], [LR], [Ko], [Va1], [Va3] to [Va5] are the most relevant literature on the existence of integral canonical models. See also [HT, §5] for a more recent translation of part of [Dr] in terms of unitary Shimura varieties. Theorems 1.4 and 1.5 can be also viewed as steps in proving Conjectures B 3.7 and B 3.12 of [Re].

In part II of this paper we will use [Va1, 6.6.2] and the formalism of smooth toroidal compactifications and elementary inductions to show that 1.6 1) always holds for $p \leq 3$ provided $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and that 1.6 2) always holds with $O(G_1, H_1, p)$ being replacement by $E(G_1, X_1)_{(p)}$ even for $p = 2$ and under no extra restriction for $p \leq 3$. This will complete the proof of Milne's conjecture of [Mi2, 2.7] and [Va1, 3.2.5] for Shimura varieties of preabelian type in mixed characteristics $(0, p)$ with $p \geq 3$ and of abelian type in mixed characteristic $(0, 2)$.

This paper brings several completely new ideas in order to essentialize, shorten and simplify [Va1] and to extend many parts of loc. cit. worked out just for $p \geq 5$ to the case of small primes $p \in \{2, 3\}$. Moreover, the projectiveness part of 1.6 corrects for $p \geq 5$ in most of the cases an error in the proof of [Va1, 3.2.3.2 ii)] which invalidated [Va1, 6.4.1.1 i) and most of 6.4.11]. Such a correction was started in [Va4, 5.2.1] and is significantly continued

here; it will be implicitly completed in part II of the paper. We would like to thank Universities of Utah and Arizona for providing us with good conditions for the writing of this paper. This research was partially supported by the NSF grant DMF 97-05376.

§2. Preliminaries

In 2.1 we include some conventions and notations. In 2.2 we first prove two extension results on reductive group schemes whose adjoints have classical Lie type and then we get a more general version of [Va1, 3.1.6]. In 2.3 we recall few basic properties of dilatations of affine group schemes over Witt rings.

2.1. Conventions and notations. Let \mathbb{F} be the algebraic closure of the field with p elements. Let k be a perfect field of characteristic p . Let $\sigma := \sigma_k$ be the Frobenius automorphism of k , $W(k)$ and $B(k) := W(k)[\frac{1}{p}]$. Let F^{sc} be the simply connected semisimple group cover of the derived group F^{der} of a reductive group scheme F . Warning: we will use the notations of 1.3 only when this is explicitly stated so; however, the standard conventions and notations of the paragraphs before 1.1 as well as of 1.1 will be used everywhere.

If M and R are as in the beginning of §1, then let

$$\mathcal{T}(M) := \bigoplus_{s,t \in \mathbb{N} \cup \{0\}} M^{\otimes s} \otimes_R M^{*\otimes t}.$$

We identify naturally $\text{End}(M) = M \otimes_R M^*$ and $\text{End}(\text{End}(M)) = M^{\otimes 2} \otimes_R M^{*\otimes 2}$. Let $x_R \in R$ be a non-divisor of 0. A family of tensors of $\mathcal{T}(M[\frac{1}{x_R}]) = \mathcal{T}(M)[\frac{1}{x_R}]$ is denoted $(w_\alpha)_{\alpha \in \mathcal{J}}$, with \mathcal{J} as the set of indices. Let M_1 be another R -module. Let $(w_{1\alpha})_{\alpha \in \mathcal{J}}$ be a family of tensors of $\mathcal{T}(M_1[\frac{1}{x_R}])$ indexed by the same set \mathcal{J} . By an isomorphism $f : (M, (w_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (M_1, (w_{1\alpha})_{\alpha \in \mathcal{J}})$ we mean an isomorphism $f : M \xrightarrow{\sim} M_1$ giving birth to an isomorphism $\mathcal{T}(M[\frac{1}{x_R}]) \xrightarrow{\sim} \mathcal{T}(M_1[\frac{1}{x_R}])$ taking w_α into $w_{1\alpha}$, $\forall \alpha \in \mathcal{J}$. If R is local, let R^{sh} be its strict henselization.

We denote two tensors or two bilinear forms in the same way provided they are obtained one from another via a reduction or a scalar extension.

2.2. Extending reductive group schemes. Let Y be a reduced scheme such that the ring of fractions η of Y is a finite product of fields. Let \mathcal{F} be a locally free sheaf on Y of finite rank. Let $GL(\mathcal{F})$ be the reductive group scheme over Y of linear automorphism of \mathcal{F} . Let U be an open subscheme of Y . Let $C := Y \setminus U$ be endowed with the reduced scheme structure. Let E_U be a reductive group scheme over U . We next recall the following result [SGA3, Vol. III, Prop. 6.1 of p. 32]:

2.2.1. Proposition. *Locally in the étale topology of U , E_U has maximal split tori.*

We refer to [Va1, 3.1.2.1 c) and 3.1.2.2 3)] for the proof of the next Proposition.

2.2.2. Proposition. *We assume that R is a DVR. Let $q : E_1 \rightarrow E_2$ be a homomorphism between flat, affine groups over R such that E_1 is a reductive group scheme and q_η is a closed embedding. Then q is a closed embedding.*

2.2.3. Theorem. *We assume Y is a regular scheme whose local rings have dimension at most 2. We also assume that C is of codimension 2 in Y and that all simple factors of pull backs of E_U^{ad} via geometric points of U are of classical Lie type. Then E_U extends uniquely to a reductive group scheme E over Y .*

Proof: It suffices to prove the Theorem under the extra assumption that Y is a local scheme $\text{Spec}(R)$ and $C = \text{Spec}(k(R))$, where $k(R)$ is the residue field of R . So η is a field. Let R_E be the \mathbb{Z} -algebra of global functions of E_U . If E exists, then E as a scheme is $\text{Spec}(R_E)$ and moreover the identity section, the multiplication operation and the inverse operation of E are uniquely determined by the corresponding analogues of E_U . This takes care of the uniqueness part. We are left to show that the morphism $\text{Spec}(R_E) \rightarrow Y$ is smooth and defines naturally a reductive group scheme over Y extending E_U . It suffices to prove this under the extra assumption that R is strictly henselian.

We show that E_U^{ab} extends to a torus over Y . Let η_1 be the smallest Galois extension of η such that $E_{\eta_1}^{\text{ab}}$ is a split torus. Let R_1 be the normalization of R in η_1 . From 2.2.1 we get that $\text{Spec}(R_1)$ has étale points above any point of Y of codimension 1. So as η_1 is a Galois extension of η , the morphism $\text{Spec}(R_1) \rightarrow Y$ is an étale cover above points of Y of codimension at most 1 and so from the classical purity theorem (see [SGA1, p. 275]) we get that it is an étale cover. So as R is strictly henselian, we have $R_1 = R$. So E_{η}^{ab} is a split torus and so also E_U^{ab} is a split torus. So E_U^{ab} extends to a split torus E^{ab} over Y .

We show that E_U^{ad} extends to an adjoint group scheme over U ; the argument for this extends until the first paragraph after Case 4 below. We write $E_{\eta}^{\text{ad}} = \prod_{i \in I} E_{i\eta}$ as a product of simple, adjoint groups over η (cf. [Ti1, 3.1.2]). Let η_i be the finite field extension of η such that $E_{i\eta} = \text{Res}_{\eta_i/\eta} E_{\eta_i}^i$, with $E_{\eta_i}^i$ as an absolutely simple, adjoint group over η_i (cf. loc. cit.). Let R_i be the normalization of R in η_i . Let V be a local ring of R which is a DVR. Let V_0 be an integral, faithfully flat, étale V -algebra such that the group $E_{V_0}^{\text{ad}}$ is split (cf. 2.2.1). Let η_0 be the field of fractions of V_0 . As the group $E_{\eta_0}^{\text{ad}} = \prod_{i \in I} \text{Res}_{\eta_i \otimes_{\eta} \eta_0 / \eta_0} E_{\eta_i \otimes_{\eta} \eta_0}^i$ is split and adjoint, we get that for each $i \in I$ the η_0 -algebra $\eta_i \otimes_{\eta} \eta_0$ is isomorphic to a product of a finite number of copies of η_0 . This implies that $R_i \otimes_R V_0$ is an étale V_0 -algebra, $\forall i \in I$. So $R_i \otimes_R V$ is an étale V -algebra, $\forall i \in I$. So as in the previous paragraph we argue that $R_i = R$, $\forall i \in I$. So $E_{i\eta}$ is an absolutely simple, adjoint group, $\forall i \in I$. Let E_{iU} be the Zariski closure of $E_{i\eta}$ in E_U^{ad} . We have a direct product decomposition $E_U = \prod_{i \in I} E_{iU}$ as one can easily check this locally in the étale topology of U .

Let S_{iU} be the split, simple, adjoint group scheme over U of the same Lie type LT_i as any geometric fibre of E_{iU} . We have a short exact sequence $0 \rightarrow S_{iU} \rightarrow \text{Aut}(S_{iU}) \rightarrow M_{iU} \rightarrow 0$, where M_{iU} is a finite, étale group scheme over U (cf. [SGA3, Vol. III, Th. 1.3 of p. 328]). It is well known that M_{iU} is $\mathbb{Z}/2\mathbb{Z}$ if $LT_i \in \{A_n | n \geq 2\} \cup \{D_n | n \geq 5\}$, is the symmetric group S_3 if $LT_i = D_4$ and is trivial otherwise. Let $\gamma_i \in H^1(\eta, \text{Aut}(S_{i\eta}))$ be the class defining $E_{i\eta}$ and let δ_i be its image in $H^1(\eta, M_{i\eta})$. The non-trivial torsors of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ over η are in one-to-one correspondence to quadratic and respectively cubic Galois extensions of η . Let η^i be the smallest Galois extension of η over which δ_i becomes the trivial class. As in the previous two paragraphs we easily get that the normalization of R in η^i is R itself. So δ_i is the trivial class. So $E_{i\eta}$ is an inner form of $S_{i\eta}$. We now show that E_{iU} extends to an adjoint group scheme over Y . We consider four Cases.

Case 1: $LT_i = A_n$. As $E_{i\eta}$ is an inner form of $S_{i\eta}$, $Z(E_{i\eta}^{\text{sc}})$ is the multiplicative group $\mu_{n+1\eta}$ over η . So $Z(E_{iU}^{\text{sc}})$ is also the multiplicative group scheme μ_{n+1U} over U . Let G_{iU} be the quotient of $\mathbb{G}_{mU} \times_U E_{iU}^{\text{sc}}$ by the μ_{n+1U} subgroup embedded via the natural embedding $\mu_{n+1U} \hookrightarrow \mathbb{G}_{mU}$ and via the composite of the inverse automorphism $\mu_{n+1U} \xrightarrow{\sim} \mu_{n+1U}$ with the natural embedding $\mu_{n+1U} \hookrightarrow E_{iU}^{\text{sc}}$. Let \mathfrak{a}_i be the locally free R -module of finite rank of global sections of the Lie sheaf $\text{Lie}(G_{iU})$ on U , cf. [FC, Lemma 6.2 of p. 181]. The group $G_{i\eta}$ is an inner form of $GL_{n+1\eta}$ and so is the group of invertible elements of a semisimple η -algebra. So $\mathfrak{a}_i \otimes_R \eta$ has a canonical structure of a semisimple η -algebra. Let Tr_i be the canonical trace form on $\mathfrak{a}_i \otimes_R \eta$. As G_{iU} locally in the étale topology of U is isomorphic to GL_{n+1U} , the η -algebra structure and the trace form on $\mathfrak{a}_i \otimes_R \eta$ extend uniquely to an R -algebra structure and respectively to a trace form Tr_i on \mathfrak{a}_i which is perfect in codimension at most 1. So the trace form Tr_i on \mathfrak{a}_i is as well perfect.

Let G_i be the group scheme over Y of invertible elements of the R -algebra \mathfrak{a}_i . As a scheme, G_i is an affine, open subscheme of the vector group scheme over Y defined by the R -module \mathfrak{a}_i . So G_i is smooth, has connected fibres and it is easy to see that $\text{Lie}(G_i)$ is \mathfrak{a}_i endowed with the Lie bracket defined by the R -algebra structure of \mathfrak{a}_i . Moreover, the notations match, i.e. the restriction of G_i to U is the group scheme G_{iU} considered in the previous paragraph. Argument: the canonical identification of $G_{i\eta}$ with the fibre of G_i over η extends to a canonical identification of G_{iU} with the restriction of G_i to U , as one can easily check this locally in the étale topology of U .

We now check that G_i is a reductive group scheme over Y . To check this it suffices to show that the fibre of G_i over the maximal point of Y is a reductive group. To check this last thing it suffices to show that $\mathfrak{a}_i \otimes_R k(R)$ is a semisimple $k(R)$ -algebra. Let \tilde{V} be an R -algebra which is a complete DVR having an algebraically closed residue field and such that $G_{i\tilde{\eta}}$ splits over the field of fractions $\tilde{\eta}$ of \tilde{V} and the natural morphism $\text{Spec}(\tilde{V}) \rightarrow Y$ has as image two distinct points, one of them being the maximal point of Y . We consider a homomorphism $G_{i\tilde{V}} \rightarrow GL(\tilde{V}^{n+1})$ over \tilde{V} which over $\tilde{\eta}$ is an isomorphism, cf. [Ja, 10.4 of Part 1]. As the trace form Tr_i is perfect, the Lie homomorphism $\mathfrak{a}_i \otimes_R \tilde{V} = \text{Lie}(G_{i\tilde{V}}) \rightarrow \text{Lie}(GL(\tilde{V}^{n+1}))$ is an isomorphism. So the \tilde{V} -homomorphism $\mathfrak{a}_i \otimes_R \tilde{V} \rightarrow \text{End}(\tilde{V}^{n+1})$ is as well an isomorphism. So $\mathfrak{a}_i \otimes_R \tilde{V}$ is a semisimple \tilde{V} -algebra. So $\mathfrak{a}_i \otimes_R k(R)$ is a semisimple $k(R)$ -algebra. So G_i is a reductive group scheme.

So G_i^{ad} is an adjoint group scheme over Y whose restriction to U is $G_{iU}^{\text{ad}} = E_{iU}$.

Case 2: $LT_i = B_n$. We consider the faithful representation W_i of $E_{i\eta}$ of dimension $2n+1$. We check that there is a free R -submodule L_i of W_i of rank $2n+1$ such that the natural monomorphism $E_{i\eta} \hookrightarrow GL(W_i)$ extends to a homomorphism $E_{iU} \rightarrow GL(L_i)_U$. If V is a local ring of R which is a DVR, then from [Ja, 10.4 of Part 1] we get the existence of a V -lattice $L_i(V)$ of W_i such that the monomorphism $E_{i\eta} \hookrightarrow GL(W_i)$ extends to a homomorphism $E_{iV} \rightarrow GL(L_i(V))$. Let \tilde{V} be a faithfully flat V -algebra which is a complete DVR having an algebraically closed residue field. So $E_{i\tilde{V}}$ is split and so there is a \tilde{V} -lattice $\tilde{L}_i(V)$ of $W_i \otimes_V \tilde{V}$ normalized by $E_{i\tilde{V}}$ and such that the special fibre of the homomorphism $E_{i\tilde{V}} \rightarrow GL(\tilde{L}_i(V))$ is an irreducible, faithful representation. So from [Ja, 10.9 of Part I] we get that the special fibre of the representation of E_{iV} on $L_i(V)$ is absolutely irreducible. So $L_i(V)$ is uniquely determined up to a $\mathbb{G}_{mR}(\eta)$ -multiple.

Let $(\text{Spec}(R[\frac{1}{f_j}]))_{j \in J}$ be a finite, affine, open cover of U such that there is a free $R[\frac{1}{f_j}]$ -

submodule $L_i(f_j)$ of W_i with the property that $\forall j \in J$ the monomorphism $E_{i\eta} \hookrightarrow GL(W_i)$ extends to a homomorphism $E_{iR[\frac{1}{f_j}]} \rightarrow GL(L_i(f_j))$ over $\text{Spec}(R[\frac{1}{f_j}])$. If $j_1, j_2 \in J$, then $L_i(f_{j_1})[\frac{1}{f_{j_2}}]$ and $L_i(f_{j_2})[\frac{1}{f_{j_1}}]$ differ just by a $\mathbb{G}_{mR}(R[\frac{1}{f_{j_1}f_{j_2}}])$ -multiple. This is so as each $L_i(V)$ is uniquely determined up to a $\mathbb{G}_{mR}(\eta)$ -multiple and as $R[\frac{1}{f_{j_1}f_{j_2}}]$ is a unique factorization domain. So as R is itself a unique factorization domain, there are non-zero elements $s_j \in \eta$ ($j \in J$) such that $s_{j_1}L_i(f_{j_1})[\frac{1}{f_{j_2}}] = s_{j_2}L_i(f_{j_2})[\frac{1}{f_{j_1}}]$, $\forall j_1, j_2 \in J$. So the free sheaves on $\text{Spec}(R[\frac{1}{f_j}])$ defined by $s_jL_i(f_j)$'s, glue together to define a locally free sheaf on U . It extends to a free sheaf on $\text{Spec}(R)$ (cf. [FC, Lemma 6.2 of p. 181]) and the global sections of it are the searched for free R -module L_i . So the monomorphism $E_{i\eta} \hookrightarrow GL(W_i)$ extends to a homomorphism $E_{iU} \rightarrow GL(L_i)_U$ which is a closed embedding, cf. 2.2.2.

Let η_{sep} be the separable closure of η . As $E_{i\eta_{\text{sep}}}$ is split, there is a non-degenerate quadratic form $F_{i\eta_{\text{sep}}}$ on $W_i \otimes_{\eta} \eta_{\text{sep}}$ such that $E_{i\eta_{\text{sep}}}$ is the subgroup of $SL(W_i \otimes_{\eta} \eta_{\text{sep}})$ preserving $F_{i\eta_{\text{sep}}}$. Argument: if η is of characteristic 2, then this follows from [Bo, 23.5 and 23.6]; the case when η is not of characteristic 2 is well known. The quadratic form $F_{i\eta_{\text{sep}}}$ is uniquely determined up to a $\mathbb{G}_{mR}(\eta_{\text{sep}})$ -multiple. Argument: any non-degenerate quadratic form on $W_i \otimes_{\eta} \eta_{\text{sep}}$ is isomorphic to the quadratic form $x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n} + x_{2n+1}^2$ on $W_i \otimes_{\eta} \eta_{\text{sep}}$ corresponding to a fixed ordered basis of $W_i \otimes_{\eta} \eta_{\text{sep}}$ (see loc. cit. for the case when η is of characteristic 2); so the statement follows from the fact that the normalizer of $E_{i\eta_{\text{sep}}}$ in $GL(W_i \otimes_{\eta} \eta_{\text{sep}})$ is generated by $E_{i\eta_{\text{sep}}}$ and by $Z(GL(W_i \otimes_{\eta} \eta_{\text{sep}}))$.

So from Hilbert's Theorem 90 we get that a non-zero scalar multiple of $F_{i\eta_{\text{sep}}}$ is definable over η , i.e. is the scalar extension of a quadratic form $F_{i\eta}$ on W_i . For any DVR V as above, a $\mathbb{G}_{mR}(\eta)$ -multiple of $F_{i\eta}$ extends to a quadratic form F_{iV} on $L_i \otimes_R V$. Argument: to check this one can work locally in the étale topology of $\text{Spec}(V)$ and so it suffices to consider the case when E_{iV} is split; but this case is trivial.

As R is a unique factorization domain, we deduce the existence of a quadratic form F_i on L_i which over each DVR V as above is a $\mathbb{G}_{mR}(V)$ -multiple of F_{iV} . The quadratic form F_i on L_i is perfect as it is perfect in codimension at most 1. So the subgroup of $SL(L_i)$ fixing F_i is an adjoint group scheme over Y extending E_{iU} .

Case 3: $LT_i = C_n$. This case is entirely the same as Case 2. We just have to replace quadratic form by alternating form and the reference to [Bo, 23.5 and 23.6] by [Bo, 23.3].

Case 4: $LT_i = D_n$. We denote also by γ_i the class of $H^1(\eta, E_{i\eta})$ whose image in $H^1(\eta, \text{Aut}(E_{i\eta}))$ is γ_i . Let \tilde{S}_{iU} be the central, isogeny cover of S_{iU} which is a split SO group. We have a central short exact sequence $0 \rightarrow \mu_2 \rightarrow \tilde{S}_{i\eta} \rightarrow S_{i\eta} \rightarrow 0$. Let $\gamma_i(2) \in H^2(\eta, \mu_2)$ be the coboundary of $\gamma_i \in H^1(\eta, S_{i\eta})$. Let $G_{i\eta}$ be the group scheme of invertible elements of the central division algebra $D_{i\eta}$ over η defining $\gamma_i(2)$. As E_{iU} is an adjoint group scheme, we easily get that $D_{i\eta}$ extends naturally to a semisimple algebra over any local ring of U which is a DVR. So $G_{i\eta}$ extends naturally to a reductive group scheme G_{iU} over U . As in Case 1 we argue that G_{iU} extends to a reductive group scheme G_i over Y . As Y is strictly henselian, G_i is split (cf. 2.2.1). So the η -algebra $D_{i\eta}$ is a matrix η -algebra and so it is η itself. So $\gamma_i(2)$ is the identity element of $H^2(\eta, \mu_2)$. So as in Case 2 we can consider the faithful representation W_i of $E_{i\eta}$ of dimension $2n$. The rest is as in Case 2 (with $2n + 1$ and the quadratic form $x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n} + x_{2n+1}^2$).

being replaced by $2n$ and respectively by the quadratic form $x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n}$. So E_{iU} extends to an adjoint group scheme over Y .

So each E_{iU} extends to an adjoint group scheme over Y . So $E_U^{\text{ad}} = \prod_{i \in I} E_{iU}$ extends to an adjoint group scheme E^{ad} over Y .

As Y is strictly henselian, E^{ad} is split (cf. 2.2.1). As E_U is isogenous to $E_U^{\text{ad}} \times_U E_U^{\text{ab}}$, the group scheme E_U is split. So the fact that E_U extends to a split, reductive group scheme E over Y follows from the uniqueness of a split reductive group scheme associated to a root datum, see [SGA3, Vol. III, Cor. 5.2 of p. 314]. We now include a second (more direct and elementary) way to check the existence of E .

Let $E_U^1 := E_U \times_{E_U^{\text{ad}} \times_U E_U^{\text{ab}}} E_U^{\text{sc}} \times_U E_U^{\text{ab}}$. So E_U^1 is an affine group scheme over U and E_U^{sc} is naturally a normal, closed subgroup of it. The quotient group E_U^1/E_U^{sc} is a group scheme of multiplicative type and so the extension of a finite, flat group scheme of finite type by a torus E_U^T . Let E_U^2 be the normal, flat, closed subgroup of E_U^1 containing E_U^{sc} and such that the quotient group E_U^2/E_U^{sc} is naturally identified with E_U^T . The group scheme E_U^2 is a reductive group scheme isomorphic to $E_U^{\text{sc}} \times_U E_U^{\text{ab}}$ and so it extends to a reductive group scheme E^2 over Y isomorphic to $E^{\text{sc}} \times_Y E^{\text{ab}}$. The kernel of the natural central epimorphism $E_U^2 \rightarrow E_U$ extends to a finite, flat, closed subgroup Z^2 of $Z(E^2)$ of multiplicative type. The quotient group scheme E^2/Z^2 is a reductive group scheme E over Y extending E_U . This ends the proof.

2.2.4. Theorem. *Under the hypotheses of 2.2.3, if E_U is a closed subgroup of $GL(\mathcal{F})_U$, then E is a closed subgroup of $GL(\mathcal{F})$.*

Proof: As in the proof of 2.2.3 we argue that to prove the Theorem it suffices to consider the case when $Y = \text{Spec}(R)$ is local and strictly henselian. As E is an affine R -scheme, we have a natural affine morphism $q : E \rightarrow GL(\mathcal{F})$ of R -schemes of finite type. It is a homomorphism as this is so generically. But from 2.2.2 we get that each fibre of q is a closed embedding and that q is a proper morphism. So q is a finite morphism. So as R is noetherian, from Nakayama's Lemma we get that q is a closed embedding. This ends the proof.

2.2.5. Proposition. *Let $m \in \mathbb{N}$. For $j \in \{1, \dots, m\}$ let $G_{j\eta}$ be a reductive subgroup of $GL(\mathcal{F})_\eta$ such that the groups $G_{j\eta}$'s commute among themselves and the direct sum $\bigoplus_{j=1}^m \text{Lie}(G_{j\eta})$ is a Lie subalgebra of $\text{Lie}(GL(\mathcal{F})_\eta)$. Let $G_{0\eta}$ be the reductive subgroup of $GL(\mathcal{F})_\eta$ generated by $G_{j\eta}$'s.*

1) *We have $\text{Lie}(G_{0\eta}) = \bigoplus_{j=1}^m \text{Lie}(G_{j\eta})$.*

2) *We assume that the Zariski closure G_j of $G_{j\eta}$ in $GL(\mathcal{F})$ is a reductive group scheme over Y , $\forall j \in \{1, \dots, m\}$. Then the Zariski closure G_0 of $G_{0\eta}$ in $GL(\mathcal{F})$ is a reductive group scheme over Y .*

Proof: Using induction on $m \in \mathbb{N}$, it suffices to prove the Proposition for $m = 2$. As 1) and 2) are local statements for the étale topology of Y , from 2.2.1 we get that it suffices to prove the Proposition under the extra assumption that G_1 and G_2 are split and Y is affine and connected. Let $C_\eta := G_{1\eta} \cap G_{2\eta}$. It is a subgroup of $G_{j\eta}$ commuting with $G_{j\eta}$, $j \in \{1, 2\}$. The Lie algebra $\text{Lie}(C_\eta)$ is included in $\text{Lie}(G_{1\eta}) \cap \text{Lie}(G_{2\eta}) = \{0\}$ and so is

trivial. So C_η is a finite, étale subgroup of $Z(G_{j\eta})$. Let C be the Zariski closure of C_η in $GL(\mathcal{F})$. It is contained in any split torus T_j of G_j .

We consider the product homomorphism $T_1 \times_Y T_2 \rightarrow GL(\mathcal{F})$. Its kernel KER is a group scheme of multiplicative type isomorphic to $T_1 \cap T_2$. The fibre over $\text{Spec}(\eta)$ of KER is isomorphic to C_η and so is a finite, flat group scheme. So KER is a finite, flat group scheme of multiplicative type. So $T_1 \cap T_2$ is a finite, flat group scheme of multiplicative type and so equal to C . We embed C in $G_1 \times_Y G_2$ via the natural embedding $C \hookrightarrow G_1$ and via the composite of the inverse isomorphism $C \xrightarrow{\sim} C$ with the natural embedding $C \hookrightarrow G_2$. We have a natural product homomorphism $q : (G_1 \times_Y G_2)/C \rightarrow GL(\mathcal{F})$ which over η can be identified naturally with the monomorphism $G_{0\eta} \hookrightarrow GL(\mathcal{F})_\eta$. But again from 2.2.2 we get that each fibre of q is a closed embedding and that q is a proper morphism. We check that q is a closed embedding. Argument: we can assume that Y is finitely generated over $\text{Spec}(\mathbb{Z})$ and so the spectrum of a noetherian \mathbb{Z} -algebra; so as in the proof of 2.2.4 we get that q is a closed embedding. So $G_0 = \text{Im}(q) \xrightarrow{\sim} (G_1 \times_Y G_2)/C$ is a reductive group scheme over Y . So 2) holds. Also as $\text{Lie}(C_\eta) = \{0\}$, we get that $\text{Lie}(G_{0\eta}) = \text{Lie}(G_{1\eta}) \oplus \text{Lie}(G_{2\eta})$. So 1) also holds. This ends the proof.

2.3. Canonical dilatations. We assume $k = \bar{k}$. Let \tilde{G} be an affine, flat group scheme over $W(k)$. Let $a \in \tilde{G}(W(k))$. The Néron measure of the defect of smoothness $\delta(a) \in \mathbb{N} \cup \{0\}$ of \tilde{G} at a is the length of the torsion part of $a^*(\Omega_{\tilde{G}/\text{Spec}(W(k))})$. As \tilde{G} is a group scheme, the value of $\delta(a)$ does not depend on a and so we denote it by $\delta(\tilde{G})$. We have $\delta(\tilde{G}) \in \mathbb{N}$ iff \tilde{G} is not smooth, cf. [BLR, Lemma of p. 65]. Let \tilde{S}_k be the Zariski closure in \tilde{G}_k of all special fibres of $W(k)$ -valued points of \tilde{G} . It is a reduced subgroup of \tilde{G}_k . We write $\tilde{S}_k = \text{Spec}(R_{\tilde{G}}/J_{\tilde{G}})$, where $\tilde{G} = \text{Spec}(R_{\tilde{G}})$ and $J_{\tilde{G}}$ is the ideal of $R_{\tilde{G}}$ defining \tilde{S}_k .

By the canonical dilatation of \tilde{G} we mean the affine \tilde{G} -scheme $\tilde{G}_1 = \text{Spec}(R_{\tilde{G}_1})$, where $R_{\tilde{G}_1}$ is the $R_{\tilde{G}}$ -subalgebra of $R_{\tilde{G}}[\frac{1}{p}]$ generated by $\frac{x}{p}$ with $x \in J_{\tilde{G}}$. The $W(k)$ -scheme \tilde{G}_1 is flat and has a canonical group structure such that the morphism $\tilde{G}_1 \rightarrow \tilde{G}$ is a homomorphism of group schemes over $W(k)$, cf. [BLR, p. 63 and (d) of p. 64]. The morphism $\tilde{G}_1 \rightarrow \tilde{G}$ has the following universal property: any morphism $\tilde{Y} \rightarrow \tilde{G}$ of flat $W(k)$ -schemes whose special fibre factors through the closed embedding $\tilde{S}_k \hookrightarrow \tilde{G}_k$, factors through $\tilde{G}_1 \rightarrow \tilde{G}$ (cf. [BLR, (b) of p. 63]). If \tilde{G} is smooth, then $\tilde{S}_k = \tilde{G}_k$ and so $\tilde{G}_1 = \tilde{G}$.

Either \tilde{G}_1 is smooth or we have $0 < \delta(\tilde{G}_1) < \delta(\tilde{G})$, cf. [BLR, Prop. 5 of p. 68]. So by using at most $\delta(\tilde{G})$ canonical dilatations (the first one of \tilde{G} , the second one of \tilde{G}_1 , etc.), we get the existence of a unique smooth, affine group scheme \tilde{G}' over $W(k)$ endowed with a homomorphism $\tilde{G}' \rightarrow \tilde{G}$ whose fibre over $B(k)$ is an isomorphism and which has the following universal property: any morphism $\tilde{Y} \rightarrow \tilde{G}$ of $W(k)$ -schemes, with \tilde{Y} a smooth $W(k)$ -scheme, factors through $\tilde{G}' \rightarrow \tilde{G}$. We refer to \tilde{G}' as the universal smoothening of \tilde{G} .

§3. On $\mathbb{Z}_{(p)}$ embeddings

Let (G_1, X_1) be a Shimura pair. Let p be a prime such that $G_{1\mathbb{Q}_p}$ is unramified. The fact that $G_{1\mathbb{Q}_p}$ is unramified is equivalent to the fact that $G_{1\mathbb{Q}_p}$ extends to a reductive group scheme $G_{1\mathbb{Z}_p}$ over \mathbb{Z}_p , cf. [Ti2, 1.10.2 and 3.8.1]. Any subgroup of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$ of the form $G_{1\mathbb{Z}_p}(\mathbb{Z}_p)$ is called hyperspecial. The hyperspecial subgroups of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$ are precisely

the compact subgroups of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$ of maximum volume with respect to the natural Haar measure on $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$, cf. [Ti2, 3.2]. Let $H_1 := G_{1\mathbb{Z}_p}(\mathbb{Z}_p)$. It is known that H_1 determines $G_{1\mathbb{Z}_p}$ (see [Ti2, 3.8.1] and [Va1, 3.1.2.1 e])). Let $G_{1\mathbb{Z}_{(p)}}$ be the unique extension of G_1 to a reductive group scheme over $\mathbb{Z}_{(p)}$ whose pull back to \mathbb{Z}_p is $G_{1\mathbb{Z}_p}$, cf. [Va1, 3.1.3]. Let v_1 be a prime of $E(G_1, X_1)$ dividing p . The prime v_1 is unramified over p , cf. [Mi3, Cor. 4.7 (a)]. So $e(v_1) = 1$. We recall from §1 that we call (G_1, X_1, H_1, v_1) a Shimura quadruple. In 3.1 to 3.3 we impose different assumptions on (G_1, X_1, H_1, v_1) and we consider different properties of (G_1, X_1, H_1, v_1) and of other similar quadruples related to it.

3.1. General properties. We consider an injective map

$$f_1 : (G, X) \hookrightarrow (G_1, X_1)$$

of Shimura pairs. Let $G_{\mathbb{Z}_{(p)}}$ be the Zariski closure of G in $G_{1\mathbb{Z}_{(p)}}$. Let $H := G(\mathbb{Q}_p) \cap H_1 = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Let v be a prime of $E(G, X)$ dividing v_1 . So v divides also p . We also view $f_1 : (G, X, H, v) \hookrightarrow (G_1, X_1, H_1, v_1)$ as an injective map between Shimura quadruples.

The functorial morphism $\mathrm{Sh}(G, X) \rightarrow \mathrm{Sh}(G_1, X_1)_{E(G, X)}$ (see [De1, 5.4]) is a closed embedding as it is so over \mathbb{C} (cf. [De1, 1.15]). So the natural morphism $\mathrm{Sh}_H(G, X) \rightarrow \mathrm{Sh}_{H_1}(G_1, X_1)_{E(G, X)}$ is a pro-finite morphism. We recall that any $O_{(v)}$ -scheme of finite type is excellent (for instance, cf. [Ma, (34.A) and (34.B)]).

3.1.1. Proposition. *We assume that $\mathrm{Sh}_{H_1}(G_1, X_1)$ has an integral canonical model \mathcal{N}_1 over $O_{(v_1)}$. We also assume that either $Z(G_1)$ is a subgroup of G or $Z(G_1)(\mathbb{Q})$ is discrete in $Z(G_1)(\mathbb{A}_f)$. Then we have the following four properties:*

- 1) *The normalization \mathcal{N}' of $\mathcal{N}_{1O_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_H(G, X)$ is a normal integral model of (G, X, H, v) having the extension property.*
- 2) *The $O_{(v)}$ -scheme \mathcal{N}' is a pro-étale cover of a normal, faithfully flat $O_{(v)}$ -scheme \mathcal{P}_1 of finite type and of relative dimension equal to $\dim_{\mathbb{C}}(X)$.*
- 3) *The morphism $\mathcal{N}' \rightarrow \mathcal{N}_{1O_{(v)}}$ is finite.*
- 4) *If (G_1, X_1) is a Siegel modular variety, then \mathcal{P}_1 is a quasi-projective $O_{(v)}$ -scheme.*

Proof: For $i \in \{1, 2\}$ let H_{0i1} be a compact, open subgroup of $G_1(\mathbb{A}_f^{(p)})$ such that \mathcal{N}_1 is a pro-étale cover of \mathcal{N}_1/H_{0i1} . Let H_{0i} be a compact, open subgroup of $G(\mathbb{A}_f^{(p)}) \cap H_{0i1}$ such that $\mathrm{Sh}(G, X)$ is a pro-étale cover of $\mathrm{Sh}_{H_{0i} \times H}(G, X)$. The morphism $\mathrm{Sh}_{H_{0i} \times H}(G, X)_{\mathbb{C}} \rightarrow \mathrm{Sh}_{H_{0i1} \times H_1}(G_1, X_1)_{\mathbb{C}}$ is of finite type as well as a formally closed embedding at each \mathbb{C} -valued point of $\mathrm{Sh}_{H_{0i} \times H}(G, X)_{\mathbb{C}}$. Let \mathcal{P}_i be the normalization of $\mathcal{N}_{1O_{(v)}}/H_{0i1}$ in $\mathrm{Sh}_{H_{0i} \times H}(G, X)$. The $O_{(v)}$ -scheme $\mathcal{N}_{1O_{(v)}}/H_{0i1}$ is faithfully flat, of finite type and so also excellent. So \mathcal{P}_i is a normal scheme finite over $\mathcal{N}_{1O_{(v)}}/H_{0i1}$ and so also a faithfully flat $O_{(v)}$ -scheme of finite type. So the relative dimension of \mathcal{P}_i over $O_{(v)}$ is $\dim_{\mathbb{C}}(\mathrm{Sh}_{H_{0i} \times H}(G, X)) = \dim_{\mathbb{C}}(X)$. The scheme \mathcal{P}_i is naturally an $\mathcal{N}_{1O_{(v)}}/H_{0i}$ -scheme.

We now take H_{01} such that it is also a normal subgroup of H_{02} . The natural morphism

$$q_{12} : \mathcal{P}_1 \rightarrow \mathcal{P}_2 \times_{\mathcal{N}_{1O_{(v)}}/H_{02}} \mathcal{N}_{1O_{(v)}}/H_{01}$$

of normal schemes is pro-finite. We now check that $q_{12E(G,X)}$ is an open closed embedding. As $q_{12E(G,X)}$ is a pro-étale cover between normal $E(G,X)$ -schemes of finite type, it is enough to show that $q_{12}(\mathbb{C})$ is injective. We have

$$\mathrm{Sh}_{H_{0i} \times H_1}(G_1, X_1)(\mathbb{C}) = G_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \backslash X_1 \times G_1(\mathbb{A}_f^{(p)}) / H_{0i} \overline{Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})},$$

where $\overline{Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}$ is the Zariski closure of $Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})$ in $Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{A}_f^{(p)})$ (cf. [Mi3, Prop. 4.11]). Also we have a natural disjoint union decomposition

$$\mathrm{Sh}_{H_{0i} \times H}(G, X)(\mathbb{C}) = \cup_{[g_j] \in G(\mathbb{Q}) \backslash G(\mathbb{Q}_p) / H} C_j \backslash X \times G(\mathbb{A}_f^{(p)}) / H_{0i},$$

where $g_j \in G(\mathbb{Q}_p)$ is a representative of the class $[g_j] \in G(\mathbb{Q}) \backslash G(\mathbb{Q}_p) / H$ and $C_j := G(\mathbb{Q}) \cap g_j H g_j^{-1}$ is independent of $i \in \{1, 2\}$. As $G_1(\mathbb{Q}_p) = G_1(\mathbb{Q}) H_1$ (cf. [Mi3, Lemma 4.9]), we can write $g_j = a_j h_j$, with $a_j \in G_1(\mathbb{Q})$ and $h_j \in H_1$. So $C_j \leq G_1(\mathbb{Q}) \cap g_j H_1 g_j^{-1} = G_1(\mathbb{Q}) \cap a_j H_1 a_j^{-1} = a_j G_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) a_j^{-1} =: C_{1j}$. Even more, we have $C_j = G(\mathbb{Q}) \cap C_{1j}$. This is so as $g_j H g_j^{-1}$ is the group of \mathbb{Z}_p -valued points of the Zariski closure of G in $a_j G_{1\mathbb{Z}_{(p)}} a_j^{-1}$.

We first show that $q_{12}(\mathbb{C})$ is injective if $Z(G_1)(\mathbb{Q})$ is discrete in $Z(G_1)(\mathbb{A}_f^{(p)})$. So $\overline{Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})} = Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})$ and so we have $\mathrm{Sh}_{H_{0i} \times H_1}(G_1, X_1)(\mathbb{C}) = G_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \backslash X_1 \times G_1(\mathbb{A}_f^{(p)}) / H_{0i}$. So to show that $q_{12}(\mathbb{C})$ is injective, it suffices to show that each one of the following commutative diagrams

$$\begin{array}{ccc} C_j \backslash X \times G(\mathbb{A}_f^{(p)}) / H_{01} & \xrightarrow{s_1} & G_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \backslash X_1 \times G_1(\mathbb{A}_f^{(p)}) / H_{01} \\ q \downarrow & & \downarrow q_1 \\ C_j \backslash X \times G(\mathbb{A}_f^{(p)}) / H_{02} & \xrightarrow{s_2} & G_{1\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \backslash X_1 \times G_1(\mathbb{A}_f^{(p)}) / H_{02}, \end{array}$$

is such that the maps q and s_1 define an injective map of $C_j \backslash X \times G(\mathbb{A}_f^{(p)}) / H_{01}$ into the fibre product of s_2 and q_1 . The vertical maps q and q_1 are the natural projections. The horizontal maps s_1 and s_2 are defined by the rule: the equivalence class $[h, g]$ is mapped into the equivalence class $[a_j^{-1} h, a_j^{-1} g]$, where $h \in X$ and $g \in G(\mathbb{A}_f^{(p)})$. So the fact that q and s_1 define an injective map of $C_j \backslash X \times G(\mathbb{A}_f^{(p)}) / H_{01}$ into the fibre product of s_2 and q_1 is an immediate consequence of the fact that $C_j = G(\mathbb{Q}) \cap C_{1j}$. So $q_{12}(\mathbb{C})$ is injective.

We now show that $q_{12}(\mathbb{C})$ is injective if $Z(G_1)$ is a subgroup of G . So $H_{0i} \overline{Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}$ is a subgroup of $G(\mathbb{A}_f^{(p)})$. So entirely as in the previous paragraph we argue that $q_{12}(\mathbb{C})$ is injective (we just need to replace “ $/H_{0i}$ ” by “ $/H_{0i} \overline{Z(G_{1\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}$ ”).

So $q_{12E(G,X)}$ is an open closed embedding. So as q_{12} is also a pro-finite morphism of normal $O_{(v)}$ -schemes of finite type, q_{12} itself is an open closed embedding. So \mathcal{P}_1 is a pro-étale cover of \mathcal{P}_2 which in characteristic 0 is an étale cover inducing Galois covers between connected components. So \mathcal{P}_1 is an étale cover of \mathcal{P}_2 inducing Galois covers between connected components. By allowing H_{01} to vary among the normal, open subgroups of H_{02} and by natural passage to limits, we get that \mathcal{N}' is a pro-étale cover of \mathcal{P}_2 and so also

that $\mathcal{P}_2 = \mathcal{N}'/H_{02}$. So the $O_{(v)}$ -scheme \mathcal{N}' together with the natural action of $G(\mathbb{A}_f^{(p)})$ on it is a normal integral model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$. It has the extension property as \mathcal{N}_1 has it. So 1) holds. As \mathcal{N}' is a pro-étale cover of \mathcal{P}_2 , 2) holds too.

As each q_{12} is an open closed embedding, by allowing H_{01} to vary through all normal, open subgroups of H_{02} we also get that \mathcal{N}' is an open closed subscheme of $\mathcal{P}_2 \times_{\mathcal{N}_1 O_{(v)}} \mathcal{N}_1 O_{(v)}$ and so is a finite $\mathcal{N}_1 O_{(v)}$ -scheme. So 3) holds.

To prove 4) we choose H_{0i2} such that \mathcal{N}_1/H_{0i2} is a moduli scheme of principally polarized abelian schemes having level N symplectic similitude structure, where $N \in \mathbb{N} \setminus \{1, 2\}$ is prime to p . Based on 2), to prove 4) it suffices to check that \mathcal{N}_1/H_{0i2} is a quasi-projective scheme over $O_{(v_1)} = \mathbb{Z}_{(p)}$. But this is implied by [MFK, Ths. 7.9 and 7.10 of p. 139]. This ends the proof.

3.1.2. Lemma. *We assume the hypotheses of 3.1.1 hold. Let \mathcal{N} be the formally smooth locus of \mathcal{N}' over $O_{(v)}$. Then \mathcal{N} is an open subscheme of \mathcal{N}' and we have $\mathcal{N}_{E(G, X)} = \mathcal{N}'_{E(G, X)}$. Moreover, if $\mathcal{N}_{k(v)}$ is a non-empty scheme, then \mathcal{N} together with the resulting action of $G(\mathbb{A}_f^{(p)})$ on it is a smooth integral model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$.*

Proof: Always \mathcal{N}' is a normal integral model of $\mathrm{Sh}_H(G, X)$ having the extension property, cf. 3.1.1 1). As \mathcal{N}' is a pro-étale cover of an excellent scheme \mathcal{P}_1 which is a normal $O_{(v)}$ -scheme of finite type (see 3.1.1 2)), \mathcal{N} is a pro-étale cover of the open subscheme \mathcal{U}_1 of \mathcal{P}_1 which is the smooth locus of \mathcal{P}_1 over $O_{(v)}$ and so \mathcal{N} is an open subscheme of \mathcal{N}' . It is well known that $\mathcal{P}_{1E(G, X)}$ is a smooth $E(G, X)$ -scheme and so we have $\mathcal{N}_{E(G, X)} = \mathcal{N}'_{E(G, X)}$. Obviously the open subscheme \mathcal{N} of \mathcal{N}' is $G(\mathbb{A}_f^{(p)})$ -invariant. So if $\mathcal{N}_{k(v)}$ is a non-empty scheme, then \mathcal{N} together with the resulting action of $G(\mathbb{A}_f^{(p)})$ on it is a smooth integral model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$. This ends the proof.

3.1.3. Lemma. *We assume that the hypotheses of 3.1.1 hold and that $e(v) \leq \max\{1, p-2\}$. Let \mathcal{N} be as in 3.1.2. Then the following three statements are equivalent:*

- (i) *the $O_{(v)}$ -scheme \mathcal{N} has the extension property;*
- (ii) *\mathcal{N} is the integral canonical model of $\mathrm{Sh}_H(G, X)$ over $O_{(v)}$;*
- (iii) *$\mathcal{N} = \mathcal{N}'$.*

Proof: As \mathcal{N} is a healthy regular scheme (cf. 1.2.1), the equivalence (i) \Leftrightarrow (ii) follows directly from 3.1.2 and def. 1.1.1 3). The implication (iii) \Rightarrow (i) follows from 3.1.1 1). The spectrum of any faithfully flat $O_{(v)}$ -algebra V which is a DVR is a healthy regular scheme. So if (i) holds, then any morphism $\mathrm{Spec}(V) \rightarrow \mathcal{N}'$ of $O_{(v)}$ -schemes factors through \mathcal{N} . This implies that any morphism $\mathrm{Spec}(V) \rightarrow \mathcal{P}_1$ of $O_{(v)}$ -schemes factors through \mathcal{U}_1 . So we have $\mathcal{U}_1 = \mathcal{P}_1$ and so $\mathcal{N} = \mathcal{N}'$. So (i) \Rightarrow (iii). So the above three statements are equivalent. This ends the proof.

3.1.4. Proposition. *We assume that the hypotheses of 3.1.1 hold, that $e(v) \leq \max\{1, p-2\}$ and that $\mathrm{Sh}_H(G, X)$ has an integral canonical model \mathcal{N} over $O_{(v)}$. Then $\mathcal{N} = \mathcal{N}'$.*

Proof: As \mathcal{N} is a healthy regular scheme (cf. def. 1.1.1 3)) and as \mathcal{N}' has the extension property, we have a natural morphism $\mathcal{N} \rightarrow \mathcal{N}'$ respecting the $G(\mathbb{A}_f^{(p)})$ -actions and extending the natural identifications $\mathcal{N}_{E(G, X)} = \mathcal{N}'_{E(G, X)} = \mathrm{Sh}_H(G, X)$. Let H_0 be a compact,

open subgroup of $G(\mathbb{A}_f^{(p)})$ such that for any inclusion $H_2 \leq H_1$ of open subgroups of H_0 , the morphisms $\mathcal{N}/H_2 \rightarrow \mathcal{N}/H_1$ and $\mathcal{N}'/H_2 \rightarrow \mathcal{N}'/H_1$ are étale covers. To end the proof it suffices to show that the induced morphism $q : \mathcal{N}/H_0 \rightarrow \mathcal{N}'/H_0$ is an isomorphism.

Let $q_V : \text{Spec}(V) \rightarrow \mathcal{N}'/H_0$ be a morphism with V a strictly henselian DVR of mixed characteristic. It lifts to a morphism $q_V^\infty : \text{Spec}(V) \rightarrow \mathcal{N}'$. As \mathcal{N} also has the extension property and as $\text{Spec}(V)$ is a healthy regular scheme, q_V^∞ factors through \mathcal{N} . So q_V factors through \mathcal{N}/H_0 . So q satisfies the valuative criterion of properness with respect to discrete valuation rings of mixed characteristic $(0, p)$. So as $q_{E(G, X)}$ is an isomorphism and as \mathcal{N}/H_0 is a flat $O_{(v)}$ -scheme, we get that q is a separated morphism of $O_{(v)}$ -schemes. From Nagata's embedding theorem (see [Na] and [Vo]) we get that \mathcal{N}/H_0 is an open, Zariski dense subscheme of a proper, normal \mathcal{N}'/H_0 -scheme $(\mathcal{N}/H_0)^c$ which is flat over $O_{(v)}$. So as q satisfies the valuative criterion of properness with respect to discrete valuation rings of mixed characteristic $(0, p)$, we get that $\mathcal{N}/H_0 = (\mathcal{N}/H_0)^c$. So q is a proper morphism of $O_{(v)}$ -schemes.

From the smoothening process of [BLR, Th. 3 of p. 61] we get the existence of an \mathcal{N}'/H_0 -scheme Z , smooth over $O_{(v)}$, quasi-projective over \mathcal{N}'/H_0 , whose fibre over $E(G, X)$ is $\mathcal{N}'_{E(G, X)}/H_0 = \text{Sh}_{H_0 \times H}(G, X)$ and such that the induced map $Z(O_{(v)}^{\text{sh}}) \rightarrow \mathcal{N}'/H_0(O_{(v)}^{\text{sh}})$ is a bijection and the smooth locus \mathcal{S} of \mathcal{N}'/H_0 over $O_{(v)}$ is naturally an open subscheme of Z . The scheme $Z \times_{\mathcal{N}'/H_0} \mathcal{N}'$ is regular, formally smooth over $O_{(v)}$ and so (cf. 1.2.1) healthy regular. So as \mathcal{N} has the extension property, $Z \times_{\mathcal{N}'/H_0} \mathcal{N}'$ is naturally an \mathcal{N} -scheme. So Z itself is naturally an \mathcal{N}/H_0 -scheme.

To show that q is an isomorphism it suffices to show that q is an isomorphism in codimension 1. We show that the assumption that q is not an isomorphism in codimension 1 leads to a contradiction. This assumption implies that there is a connected component $\mathcal{C}_{k(v)}$ of $\mathcal{N}_{k(v)}/H_0$ dominating a reduced closed subscheme Z_0 of $\mathcal{N}'_{k(v)}/H_0$ of dimension $d < \dim(\mathcal{C}_{k(v)})$. As \mathcal{S} is an open subscheme of Z and so an \mathcal{N}/H_0 -scheme, the subscheme Z_0 of \mathcal{N}'/H_0 is a closed subscheme of the complement of \mathcal{S} in \mathcal{N}'/H_0 . Let \mathcal{C} be the open subscheme of \mathcal{N}/H_0 defined by $\mathcal{C}_{k(v)}$ and the generic fibre of \mathcal{N}/H_0 . The morphism $\mathcal{C} \rightarrow \mathcal{N}'/H_0$ lifts to a morphism $\tilde{q} : \mathcal{C} \rightarrow \tilde{Z}$, where \tilde{Z} is obtained from \mathcal{N}'/H_0 through the first dilatation needed to get Z . Argument: this first dilatation is obtained by blowing up the maximal reduced closed subscheme Z_{00} of $\mathcal{N}'_{k(v)}/H_0$ having the property that it is included in $\mathcal{N}'/H_0 \setminus \mathcal{S}$ and the points of it with values in the residue field \mathbb{F} of $O_{(v)}^{\text{sh}}$ and which admit lifts (in \mathcal{N}'/H_0) to O^{sh} -valued points, are Zariski dense in it (cf. [BLR, p. 72]); so as Z_0 is a closed subscheme of Z_{00} , the existence of \tilde{q} is implied by [BLR, Prop. 1 of p. 63]. We know that \tilde{Z} is an affine \mathcal{N}'/H_0 -scheme, cf. the properties of dilatations [BLR, p. 62]. So as \mathcal{C} is smooth and as its fibres over points of \mathcal{N}'/H_0 are proper schemes, \tilde{q} dominates a closed subscheme \tilde{Z}_0 of $\tilde{Z}_{k(v)}$ of dimension d . As above we argue that \tilde{Z}_0 is included in the non-smooth locus of \tilde{Z} . By induction on the number of dilatations needed to construct Z from \mathcal{N}'/H_0 , we get that \mathcal{C} is a Z -scheme and that the image of $\mathcal{C}_{k(v)}$ in Z is of dimension d . So the composite morphism $\mathcal{C} \rightarrow Z \rightarrow \mathcal{N}/H_0$ is not an open embedding. Contradiction. So q is an isomorphism. This ends the proof of the Proposition.

3.1.5. Example. With the notations of 1.3, we take (G_1, X_1) and \mathcal{N}_1 to be $(GSp(W, \psi), S)$ and respectively \mathcal{M} . So the definitions of \mathcal{N}' in 1.3 and 3.1.1 coincide. Always, the rank

1 split torus $Z(GSp(W, \psi))$ is a subtorus of G . So the hypotheses of 3.1.1 hold. So 3.1.1 and 3.1.2 apply entirely. In particular, \mathcal{N} is always a $G(\mathbb{A}_f^{(p)})$ -invariant, open subscheme of \mathcal{N}' . If moreover we have $e(v) \leq \max\{1, p-2\}$, then 3.1.3 and 3.1.4 also apply.

3.2. Standard PEL situations. In this section let $L, L_{(p)}, f : (G, X) \hookrightarrow (GSp(W, \psi), S)$, $G_{\mathbb{Z}_{(p)}}$, $E(G, X)$, v , $k(v)$, $O_{(v)}$, \mathcal{N}' and \mathcal{N} be as in 1.3. In this section we assume that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and that there is a $\mathbb{Z}_{(p)}$ -subalgebra $\mathcal{B}_{(p)}$ of $\text{End}(L_{(p)})$ which over $W(\mathbb{F})$ is a product of matrix algebras, which is self dual with respect to ψ and is such that G is the identity component of the subgroup of $GSp(W, \psi)$ fixing the elements of $\mathcal{B}_{(p)}$. We recall that in this case $\text{Sh}(G, X)$ is called a Shimura variety of PEL type (see [Ko], etc.). We also refer to the triple (f, L, v) as a standard PEL situation.

3.2.1. Theorem. *Then $\mathcal{N} = \mathcal{N}'$ is an integral canonical model of $\text{Sh}_H(G, X)$ over $O_{(v)}$.*

Proof: As $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, the group $G_{\mathbb{Q}_p}$ is unramified and so from [Mi3, Cor. 4.7 (a)] we get $e(v) = 1$. So based on 3.1.3 and 3.1.5, it suffices to show that $\mathcal{N}' = \mathcal{N}$ and so that \mathcal{N}' is formally smooth over $O_{(v)}$. The case $p > 2$ is well known (see [Zi], [LR], [Ko, §5] and [Va1, 4.3.11 and 5.6.3]). We refer to [Va3, 1.3] for the case $p = 2$. This ends the proof.

3.3. Proposition. *We assume (G_1, X_1) is a simple, adjoint Shimura pair of abelian type. Then there is an injective map $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$ and a $\mathbb{Z}_{(p)}$ -lattice $L_{(p)}$ of W on which ψ induces a perfect alternating form, such that the following four properties hold:*

- (i) *we have $(G^{\text{ad}}, X^{\text{ad}}) = (G_1, X_1)$;*
- (ii) *the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $GL(L_{(p)})$ is a reductive group scheme whose adjoint is the adjoint group $G_{1\mathbb{Z}_{(p)}}$ introduced in the beginning of §3;*
- (iii) *if (G'_1, X'_1) is a Shimura pair of abelian type having (G_1, X_1) as its adjoint, then G^{der} is an isogeny cover of G_1^{der} (i.e. G^{der} is the maximal isogeny cover of G_1 allowed by the abelian types);*
- (iv) *if either (G_1, X_1) is of A_n type or (G_1, X_1) is of C_n or $D_n^{\mathbb{H}}$ type and moreover (G_1, X_1) has no compact factors, then there is a $\mathbb{Z}_{(p)}$ -subalgebra $\mathcal{B}_{(p)}$ of $\text{End}(L_{(p)})$ as in 3.2 (i.e. (f, L, v) is a standard PEL situation, for any \mathbb{Z} -lattice L of W on which ψ induces a perfect alternating form and which has the property that $L_{(p)} = L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$).*

Proof: If (G_1, X_1) is of A_n type, then this is proved in [Va4, Prop. 4.1]. Let now (G_1, X_1) be of B_n or C_n type with $n \geq 2$ or of $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type with $n \geq 4$. If $p \geq 3$, then the Proposition is just a weaker form of [Va4, Th. 4.8 and Rm. 4.8.1 4)]. We now recall a slight variant of loc. cit. which is closer in spirit to [Va1, 6.5 and 6.6] and which has two extra features which will play key roles in §6. We work with an arbitrary prime $p \geq 2$.

Let F_1 be a totally real number field such that G_1 is $\text{Res}_{F_1/\mathbb{Q}} \tilde{G}_1$, with \tilde{G}_1 as an absolutely simple, adjoint group over F_1 (cf. [De2, 2.3.4 (a)]). We write

$$F_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I_p} F_{1i}$$

as a product of p -adic fields. As $G_{1\mathbb{Z}_p}$ is a reductive group scheme, it splits over an unramified extension of \mathbb{Z}_p (see 2.2.1). So $G_{1\mathbb{Q}_p}$ splits over an unramified, finite field extension F_{10} of \mathbb{Q}_p . So each F_{1i} is a subfield of F_{10} . So F_1 is unramified over p . Let F_2 be a totally real number field containing F_1 , unramified over p and such that we can write

$$F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I_p} F_{2i}$$

as a product of p -adic fields indexed by the set I_p and with each F_{1i} as a subfield of F_{2i} .

An element $i \in I_p$ will be called compact if the following property holds:

(v) *the extension of the normal subgroup $\text{Res}_{F_{1i}/\mathbb{Q}_p} \tilde{G}_{F_{1i}}$ of $G_{1\mathbb{Q}_p}$ to \mathbb{C} via an (any) $O_{(v_1)}$ -embedding $W(k(v_1)) \hookrightarrow \mathbb{C}$, is such that it has simple factors which are pull backs to $\text{Spec}(\mathbb{C})$ of compact factors of $G_{1\mathbb{R}}$.*

The set I_p has compact elements iff (G_1, X_1) has compact factors. Let E_2 be a totally imaginary quadratic extension of F_2 unramified over p . If (G_1, X_1) has no compact factors, then we take $F_2 = F_1$. If (G_1, X_1) has compact factors, then we take F_2 and E_2 such that the following two additional properties hold:

(vi) *there is a compact element $i_0 \in I_p$ such that we have $F_{1i} = F_{2i}$ for all $i \in I_p \setminus \{i_0\}$; so we have $[F_2 : F_1] = [F_{2i_0} : F_{1i_0}]$;*

(vii) *for any compact element $i \in I_p$ the field $E_2 \otimes_{F_2} F_{2i}$ is the unramified quadratic field extension of F_{2i} .*

Let $\tilde{G}_{1F_{1(p)}}$ be the adjoint group scheme over $F_{1(p)}$ extending \tilde{G}_1 and such that we have $G_{1\mathbb{Z}_{(p)}} = \text{Res}_{F_{1(p)}/\mathbb{Z}_{(p)}} \tilde{G}_{1F_{1(p)}}$. One constructs $\tilde{G}_{1F_{1(p)}}$ as a direct factor of $G_{1F_{1(p)}}$. Let F_3 be a number field containing F_1 , unramified over p and such that a maximal torus $T_{1F_{1(p)}}$ of $\tilde{G}_{1F_{1(p)}}$ splits over $F_{3(p)}$. Let F_4 be the composite field of F_3 and F_2 . Warning: the number fields F_3 and F_4 are not in general totally real number fields.

Let $G_{2\mathbb{Z}_{(p)}} := \text{Res}_{F_{2(p)}/\mathbb{Z}_{(p)}} \tilde{G}_{1F_{2(p)}}$. Let $G_2 := G_{2\mathbb{Q}} = \text{Res}_{F_2/\mathbb{Q}} \tilde{G}_{1F_2}$. Let (G_2, X_2) be the adjoint Shimura pair such that the natural monomorphism $G_1 = \text{Res}_{F_1/\mathbb{Q}} \tilde{G}_1 \hookrightarrow G_2 = \text{Res}_{F_2/\mathbb{Q}} \tilde{G}_{1F_2}$ extends to an injective map $(G_1, X_1) \hookrightarrow (G_2, X_2)$ of Shimura pairs (it exists as F_2 is totally real).

See [Bou, planche II to IV] for the standard notations of weights associated to the B_n , C_n and D_n Lie types. We consider the representation $\rho_0 : \tilde{G}_{1F_{4(p)}}^{\text{sc}} \rightarrow GL(L_{1(p)})$ over $F_{4(p)}$ such that the following properties hold:

(viii) *$L_{1(p)}$ is a free $F_{4(p)}$ -module of finite rank;*

(ix) *the representation ρ_0 is a direct sum of representations associated to the following minuscule weights: ϖ_n if (G_1, X_1) is of B_n type, ϖ_1 if (G_1, X_1) is of C_n type, ϖ_1 if (G_1, X_1) is of $D_n^{\mathbb{H}}$ type, and ϖ_{n-1} and ϖ_n if (G_1, X_1) is of $D_n^{\mathbb{R}}$ type.*

Here the weights are with respect to the inverse image in $\tilde{G}_{1F_{4(p)}}^{\text{sc}}$ of the split torus $T_{1F_{4(p)}}$ of $\tilde{G}_{1F_{4(p)}}$. Warning: if \tilde{G}_1 is of D_4 Dynkin type, then we choose the numbering of weights as in [De2, p. 272]. The representation ρ_0 is a closed embedding iff (G_1, X_1) is of

B_n , C_n or $D_n^{\mathbb{R}}$ type. If (G_1, X_1) is of $D_n^{\mathbb{H}}$ type, then ρ_0 is the composite of an isogeny of degree 2 with a monomorphism which is a closed embedding.

Let

$$L_{(p)} := E_{2(p)} \otimes_{F_{2(p)}} L_{1(p)}$$

but viewed as a $\mathbb{Z}_{(p)}$ -module and not as an $E_{2(p)}$ -module; it is naturally a $G_{2\mathbb{Z}_{(p)}}^{\text{sc}}$ -module and so also a $G_{1\mathbb{Z}_{(p)}}^{\text{sc}}$ -module. Let $W := L_{(p)}[\frac{1}{p}]$; it is a \mathbb{Q} -vector space on which G_2^{sc} and G_1^{sc} act naturally. Let G^{der} (resp. \tilde{G}^{der}) be the semisimple subgroup of $GL(W)$ which is the image of G_1^{sc} (resp. of G_2^{sc}) in $GL(W)$. The Zariski closure of G^{der} (resp. \tilde{G}^{der}) in $GL(L_{(p)})$ is a semisimple subgroup $G_{\mathbb{Z}_{(p)}}^{\text{der}}$ (resp. $\tilde{G}_{\mathbb{Z}_{(p)}}^{\text{der}}$) of $GL(L_{(p)})$ having $G_{1\mathbb{Z}_{(p)}}$ (resp. $G_{2\mathbb{Z}_{(p)}}$) as its adjoint. As $L_{(p)}$ has also a natural structure of an $E_{2(p)}$ -module, the torus $\text{Res}_{E_{2(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{mE_{2(p)}}$ acts naturally and faithfully on $L_{(p)}$.

Let $T_{\mathbb{Z}_{(p)}}$ be the torus of $GL(L_{(p)})$ generated by $Z(GL(L_{(p)}))$ and by the maximal subtorus $T_{\mathbb{Z}_{(p)}}^c$ of $\text{Res}_{E_{2(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{mE_{2(p)}}$ which over \mathbb{R} is compact. The torus $T_{\mathbb{Z}_{(p)}}^c$ is isogenous to the quotient torus $\text{Res}_{E_{2(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{mE_{2(p)}} / \text{Res}_{F_{2(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{mF_{2(p)}}$ and so has rank $[F_2 : \mathbb{Q}]$. Let G (resp. \tilde{G}) be generated by a subtorus $T'_{\mathbb{Q}}$ of $T_{\mathbb{Q}}$ and by G^{der} (resp. and by \tilde{G}^{der}). The notations match, i.e. the derived group of G is indeed the group G^{der} introduced in the previous paragraph. Moreover, the Zariski closure of $T'_{\mathbb{Q}} = Z^0(G)$ in $GL(L_{(p)})$ is a subtorus of $T_{\mathbb{Z}_{(p)}}$. So from 2.2.5 2) we get that the Zariski closures $G_{\mathbb{Z}_{(p)}}$ and $\tilde{G}_{\mathbb{Z}_{(p)}}$ of G and respectively \tilde{G} in $GL(L_{(p)})$ are reductive group schemes. Their adjoints are $G_{1\mathbb{Z}_{(p)}}$ and respectively $G_{2\mathbb{Z}_{(p)}}$. So (ii) holds. Using a standard embedding $SL_{m\mathbb{Z}_{(p)}} \hookrightarrow Sp_{2m\mathbb{Z}_{(p)}}$ with $m := \dim_{\mathbb{Q}}(W)$ and replacing if needed $L_{1(p)}$ by $L_{1(p)} \oplus L_{1(p)}^*$, we can assume that there is a perfect alternating form $\tilde{\psi}$ on $L_{(p)}$ normalized by $\tilde{G}_{\mathbb{Z}_{(p)}}$.

In [De2, proof of 2.3.10] it is shown that we can choose the torus $T'_{\mathbb{Q}}$ (for instance, we can take $T'_{\mathbb{Q}}$ to be $T_{\mathbb{Q}}$ itself) such that there is a monomorphism $x : \mathbb{S} \hookrightarrow \tilde{G}_{\mathbb{R}}$ having the following three properties:

- (x) the Hodge \mathbb{Q} -structure on W defined by x is of type $\{(-1, 0), (0, -1)\}$;
- (xi) there is a non-degenerate alternating form ψ on W normalized by \tilde{G} and such that $2\pi i\psi$ is a polarization of the Hodge \mathbb{Q} -structure on W mentioned in property (x);
- (xii) if \tilde{X} is the $\tilde{G}(\mathbb{R})$ -conjugacy class of x , then we get a Shimura pair (\tilde{G}, \tilde{X}) whose adjoint is (G_2, X_2) .

The group $G_{2\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ permutes transitively the connected components of X_2 , cf. [Va1, 3.3.3]. So by replacing x by its inner conjugate under an element of $\tilde{G}(\mathbb{R})G_{2\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ and by replacing if needed ψ by another non-degenerate alternating form on W such that (xi) holds (cf. [De2, 2.3.3]), we can choose x such that its image in X_2 (i.e. the composite of x with the natural epimorphism $\tilde{G}_{\mathbb{R}} \twoheadrightarrow \tilde{G}_{\mathbb{R}}^{\text{ad}} = G_{2\mathbb{R}}$) belongs to X_1 . So x factors through $G_{\mathbb{R}}$ and so we can view x also as a monomorphism $x : \mathbb{S} \hookrightarrow G_{\mathbb{R}}$.

Let ALT be the free $\mathbb{Z}_{(p)}$ -module of alternating forms on $L_{(p)}$ normalized by $\tilde{G}_{\mathbb{Z}_{(p)}}$. So $\tilde{\psi} \in ALT$ and $\psi \in ALT[\frac{1}{p}]$. Based on [De2, 1.1.18 b)] and on standard arguments of inequivalent valuations, we get that we can choose ψ such that we also have $\psi \in ALT$ and moreover ψ and $\tilde{\psi}$ are congruent mod p . So ψ induces a perfect alternating form on $L_{(p)}$.

Let X be the $G(\mathbb{R})$ -conjugacy class of $x : \mathbb{S} \hookrightarrow G_{\mathbb{R}}$. From (xii) we get that the pair (G, X) is a Shimura pair. As x maps into an element of X_1 , the adjoint of (G, X) is (G_1, X_1) . So (i) holds. We already checked that (ii) holds. The fact that (iii) holds is obvious if (G_1, X_1) is of B_n , C_n or $D_n^{\mathbb{R}}$ (as G^{der} is simply connected) and is implied by [De2, 2.3.8 and 2.3.10 (ii)] if (G_1, X_1) is of $D_n^{\mathbb{H}}$ type.

Let now (G_1, X_1) be of C_n or $D_n^{\mathbb{H}}$ (resp. of B_n or $D_n^{\mathbb{R}}$) type. Let $C_{\mathbb{Z}_{(p)}}$ be the centralizer of $G_{\mathbb{Z}_{(p)}}$ in $GL(L_{(p)})$. The representation of $G_{W(\mathbb{F})}$ on $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ is a direct sum of irreducible representations of rank $2n$ (resp. of rank 2^n or 2^{n-1} and which are either spin or half spin representations and) on which a product factor of $G_{W(\mathbb{F})}^{\text{der}}$ isomorphic to $Sp_{2nW(\mathbb{F})}$ or $SO_{2nW(\mathbb{F})}$ (resp. isomorphic to $\text{Spin}_{2n+1W(\mathbb{F})}$ or to a quotient of $\text{Spin}_{2nW(\mathbb{F})}$ by $\mu_{2W(\mathbb{F})}$) acts non-trivially. So the special fibres of any two such irreducible representations are also irreducible (see [Va5, Lemma of the proof of 7.8.3.1]) and moreover are isomorphic iff the two representations over $W(\mathbb{F})$ are isomorphic. This implies that $C_{W(\mathbb{F})}$ is a reductive group scheme. So $C_{\mathbb{Z}_{(p)}}$ is a reductive group scheme. Let $\mathcal{B}_{(p)}$ be the semisimple $\mathbb{Z}_{(p)}$ -subalgebra of $\text{End}(L_{(p)})$ such that as $\mathbb{Z}_{(p)}$ -modules we have $\mathcal{B}_{(p)} = \text{Lie}(C_{\mathbb{Z}_{(p)}})$.

If (G_1, X_1) is of C_n or $D_n^{\mathbb{H}}$ type and has no compact factors, then we can choose $T'_{\mathbb{Q}}$ to be $Z(GL(W))$ (cf. [De2, Rm. 2.3.13]) and so the centralizer D of $\mathcal{B}_{(p)}[\frac{1}{p}]$ in $GL(W)$ is a reductive group such that $D_{\mathbb{C}}$ is isomorphic to $GL_{2n\mathbb{C}}^{[F_1:\mathbb{Q}]}$. Moreover the monomorphism $G_{\mathbb{C}}^{\text{der}} \rightarrow D_{\mathbb{C}}^{\text{der}}$ is a product of $[F_1 : \mathbb{Q}]$ -monomorphism of the form $Sp_{2n\mathbb{C}} \hookrightarrow SL_{2n\mathbb{C}}$ or of the form $SO_{2n\mathbb{C}} \hookrightarrow SL_{2n\mathbb{C}}$ and the representation on $W \otimes_{\mathbb{Q}} \mathbb{C}$ of each direct factor $SL_{2n\mathbb{C}}$ of $D_{\mathbb{C}}^{\text{der}}$ is isotypic. So the Lie algebra of each such direct factor is normalized without being fixed by the involution of $\text{End}(W \otimes_{\mathbb{Q}} \mathbb{C})$ defined by ψ . We easily get that $\text{Lie}(G)$ is the Lie algebra of the identity component \tilde{G}' of the centralizer of $\mathcal{B}_{(p)}[\frac{1}{p}]$ in $GSp(W, \psi, S)$. So the natural monomorphism $G \hookrightarrow \tilde{G}'$ is an isomorphism. So (iv) holds. This ends the proof.

3.3.1. Remark. We refer to the proof of 3.3. Always $\mathcal{B}_{(p)}$ is an $F_{1(p)}$ -algebra and the action of $F_{1(p)}$ on $L_{(p)}$ has no fixed non-trivial subspaces. The centralizers of $T_{\mathbb{Z}_{(p)}}$ and $\text{Res}_{E_{2(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{mE_{2(p)}}$ in $GL(L_{(p)})$ are the same. So if we choose $T'_{\mathbb{Q}}$ to be $T_{\mathbb{Q}}$, then $\mathcal{B}_{(p)}$ is an $E_{2(p)}$ -algebra and so also an $F_{2(p)}$ -algebra.

3.3.2. Lemma. *We refer to the proof of 3.3, with (G_1, X_1) of B_n or C_n type with $n \geq 1$ or of $D_n^{\mathbb{H}}$ type with $n \geq 4$. Let m be $2n$ if (G_1, X_1) is of C_n or $D_n^{\mathbb{H}}$ type and be 2^n if (G_1, X_1) is of B_n type. We take $T'_{\mathbb{Q}}$ to be $T_{\mathbb{Q}}$. Then $Z(G) = Z^0(G)$ and the centralizer of $\mathcal{B}_{(p)}$ in $GSp(L_{(p)}, \psi)$ is a reductive group scheme $\tilde{G}'_{\mathbb{Z}_{(p)}}$ having $Z(G_{\mathbb{Z}_{(p)}})$ as its center and having a derived group which over $W(\mathbb{F})$ is isomorphic to $SL_{mW(\mathbb{F})}^{[F_2:\mathbb{Q}]}$.*

Proof: The torus $T_{W(\mathbb{F})}^c$ is a product $\prod_{j=1}^{[F_2:\mathbb{Q}]} T_j$ of rank 1 tori such that the representation of $T_{W(\mathbb{F})}^c$ on $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ is a direct sum $\bigoplus_{j=1}^{2[F_2:\mathbb{Q}]} L_j$ having the following property:

(*) for any $j \in \{1, \dots, [F_2 : \mathbb{Q}]\}$, the torus T_j acts via the identical character on L_j , via the inverse of the identical character on $L_{j+[F_2:\mathbb{Q}]}$ and trivially on L_{j_1} , $\forall j_1 \in \{1, \dots, 2[F_2 : \mathbb{Q}]\} \setminus \{j, j + [F_2 : \mathbb{Q}]\}$.

As $T_{W(\mathbb{F})}^c$ fixes ψ , for $j_1, j_2 \in \{1, \dots, 2[F_2 : \mathbb{Q}]\}$ we have

(**) $\psi(L_{j_1}, L_{j_2}) \neq \{0\}$ iff $|j_2 - j_1| = [F_2 : \mathbb{Q}]$.

The derived group $G_{W(\mathbb{F})}^{\text{der}}$ acts on each L_j via a direct factor DF_j of it isomorphic to an $Sp_{2nW(\mathbb{F})}$, an $SO_{2nW(\mathbb{F})}$ or an $\text{Spin}_{2n+1W(\mathbb{F})}$ group. Moreover, the representation of DF_j on L_j is a direct sum of isomorphic representations associated to a unique minuscule weight and which shows up in 3.3 (ix). This implies that $Z(G_{W(\mathbb{F})}^{\text{der}}) \simeq \mu_2^{[F_1 : \mathbb{Q}]}$ is a subgroup of $T_{W(\mathbb{F})}^c$. So $Z(G_{W(\mathbb{F})}) = Z^0(G_{W(\mathbb{F})})$ and so $Z(G) = Z^0(G)$.

The group $C_{W(\mathbb{F})}$ is a product $\prod_{j=1}^{2[F_2 : \mathbb{Q}]} C_j$, with C_j a $GL_{mW(\mathbb{F})}$ group acting trivially on each L_{j_1} with $j_1 \in \{1, \dots, 2[F_2 : \mathbb{Q}]\} \setminus \{j\}$. The representation of C_j on L_j is a direct sum of standard rank m representations. Due to (**), the intersection $C_{W(\mathbb{F})} \cap Sp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}), \psi)$ is a product $\prod_{j=1}^{[F_2 : \mathbb{Q}]} D_j$, with D_j as a $GL_{mW(\mathbb{F})}$ group having T_j as its center. So the centralizer of $\mathcal{B}_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ in $GSp(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}), \psi)$ is a reductive group scheme having $Z(G_{W(\mathbb{F})})$ as its center and having a derived group isomorphic to $SL_{mW(\mathbb{F})}^{[F_2 : \mathbb{Q}]}$. So the centralizer $\tilde{G}'_{\mathbb{Z}_{(p)}}$ of $\mathcal{B}_{(p)}$ in $GSp(L_{(p)}, \psi)$ is a reductive group scheme such that its center is $Z(G_{\mathbb{Z}_{(p)}})$ and $\tilde{G}'_{W(\mathbb{F})}^{\text{der}}$ is isomorphic to $SL_{mW(\mathbb{F})}^{[F_2 : \mathbb{Q}]}$. This ends the proof.

3.3.3. Remark. We refer to the proof of 3.3, with (G_1, X_1) of $D_n^{\mathbb{R}}$ type with $n \geq 4$. If we choose $T'_{\mathbb{Q}}$ to be $T_{\mathbb{Q}}$, then entirely as in the proof of 3.3.2 we argue that the centralizer of $\mathcal{B}_{(p)}$ in $GSp(L_{(p)}, \psi)$ is a reductive group scheme $\tilde{G}'_{\mathbb{Z}_{(p)}}$ whose derived group is such that its extension to $W(\mathbb{F})$ is isomorphic to $SL_{2^{n-1}W(\mathbb{F})}^{2[F_2 : \mathbb{Q}]}$. The only difference is that this time we have two minuscule weights ϖ_{n-1} and ϖ_n (see 3.3 (ix)) and the representations associated to them are half spin representations and so are of rank 2^{n-1} ; this justifies the double number $2[F_2 : \mathbb{Q}]$ of copies of $SL_{2^{n-1}W(\mathbb{F})}$.

§4. Crystalline applications

In 4.1 we recall the simplest variant of the main result of [dJ2]. In 4.2 we first introduce several notations needed to prove 1.4 to 1.6 and then we apply the main result of [Va6]. In 4.3 we prove 1.4. In 4.4 we present a simple criterion on when the special fibre $\mathcal{N}_{k(v)}$ of the $\mathcal{O}_{(v)}$ -scheme \mathcal{N} of 1.3 is non-empty. In 4.5 we apply 1.4 1) and 4.4 to get a solution to Langlands' conjecture of [La, p. 411] for the case of Shimura varieties of Hodge type. In 4.6 we refine the deformation theories of [Fa, §7] and [Va1, 5.4] as allowed by [Va6].

For (crystalline or de Rham) Fontaine comparison theory we refer to [Fo]; see also [Fa, §5] and [Va6]. Let k be as in 2.1. Let x be an independent variable. As the Verschiebung maps of p -divisible groups will not be mentioned at all in what follows, we will use the terminology F -crystals (resp. filtered) F -crystals associated to p -divisible groups over k , $k[[x]]$ or $k((x))$ (resp. over $W(k)$ or $W(k)[[x]]$) instead of the terminology Dieudonné (resp. filtered Dieudonné) F -crystals used in [BBM, Ch. 3], [BM, Ch. 2 and 3] or [dJ1].

The simplest form of [dJ2, 1.1] says:

4.1. Theorem (de Jong). *The natural functor from the category of non-degenerate F -crystals over $\text{Spec}(k[[x]])$ to the category of non-degenerate F -crystals over $\text{Spec}(k((x)))$ is fully faithful.*

For the notion non-degenerate crystal [dJ2] refers to [Saa, 3.1.1 of p. 331]. For us the only important thing will be that any F -crystal of an abelian scheme over $\mathrm{Spec}(k[[x]])$ is non-degenerate and that the non-degenerate F -crystals are stable under tensor products.

4.2. A setting. Warning: from now on until the end of the paper, the field k will be assumed to be algebraically closed and the following notations L , $L_{(p)}$, $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$, $E(G, X)$, v , $k(v)$, $O_{(v)}$, $K_p = GSp(L, \psi)(\mathbb{Z}_p)$, $G_{\mathbb{Z}_{(p)}}$, $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, \mathcal{M} , \mathcal{N}' , \mathcal{N} , $(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ will be as in 1.3.

We will also use the terminology of [De3] on Hodge cycles on an abelian scheme A_Z over a reduced \mathbb{Q} -scheme Z . So we write each Hodge cycle v on A_Z as a pair $(v_{dR}, v_{\acute{e}t})$, where v_{dR} and $v_{\acute{e}t}$ are the de Rham and respectively the étale component of v . The étale component $v_{\acute{e}t}$ as its turn has an l -component $v_{\acute{e}t}^l$, for any rational prime l . For instance, if Z is the spectrum of a field E , then $v_{\acute{e}t}^p$ is a suitable $\mathrm{Gal}(\overline{E}/E)$ -invariant tensor of the tensor algebra of $H_{\acute{e}t}^1(A_{\overline{Z}}, \mathbb{Q}_p) \oplus (H_{\acute{e}t}^1(A_{\overline{Z}}, \mathbb{Q}_p))^* \oplus \mathbb{Q}_p(1)$, where $\overline{Z} := \mathrm{Spec}(\overline{E})$ and $\mathbb{Q}_p(1)$ is the usual Tate twist. If moreover \overline{E} is a subfield of \mathbb{C} , then we also use the Betti realization of v : it corresponds to v_{dR} (resp. to $v_{\acute{e}t}^l$) via the canonical isomorphism relating the Betti cohomology with \mathbb{Q} -coefficients of $A_Z \times_Z \mathrm{Spec}(\mathbb{C})$ with the de Rham (resp. the \mathbb{Q}_l étale) cohomology of $A_{\overline{Z}}$ (see [De3, §2]).

Let TR be the trace form on $\mathrm{End}(W)$. If K is a field of characteristic 0 and if G_0 is a reductive subgroup of $GL(W \otimes_{\mathbb{Q}} K)$, then the restriction TR_0 of TR to $\mathrm{Lie}(G_0)$ is non-degenerate. Argument: it suffices to consider the cases when $K = \overline{K}$ and G_0 is either \mathbb{G}_{mK} or a semisimple group whose adjoint is simple; the fact that TR_0 is non-degenerate is trivial in the first case and it follows from Cartan's solvability criterion in characteristic 0 in the second case. Let π_{G_0} be the projector of $\mathrm{End}(W \otimes_{\mathbb{Q}} K)$ on $\mathrm{Lie}(G_0)$ along the perpendicular of $\mathrm{Lie}(G_0)$ with respect to TR . If G_K normalizes G_0 , then G_K fixes π_{G_0} .

As $Z(GL(W))$ is a subgroup of G , each tensor of $\mathcal{T}(W^*)$ fixed by G belongs to $\oplus_{u \in \mathbb{N} \cup \{0\}} W^{*\otimes u} \otimes_{\mathbb{Q}} W^{\otimes u}$. So as G is a reductive group, from [De3, 3.1 c)] we get the existence of a family $(v_{\alpha})_{\alpha \in \mathcal{J}}$ of tensors in spaces of the form $W^{*\otimes u} \otimes_{\mathbb{Q}} W^{\otimes u} \subset \mathcal{T}(W^*)$ for some $u \in \mathbb{N} \cup \{0\}$, which contains π_G and has the property that G is the subgroup of $GL(W)$ fixing v_{α} , $\forall \alpha \in \mathcal{J}$. The choice of L and $(v_{\alpha})_{\alpha \in \mathcal{J}}$ allows a moduli interpretation of $\mathrm{Sh}(G, X)$ (see [De1], [De2], [Mi3] and [Va1, 4.1 and 4.1.3]). For instance, $\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$ is the set of isomorphism classes of principally polarized abelian varieties over \mathbb{C} of dimension $\frac{\dim_{\mathbb{Q}}(W)}{2}$, carrying a family of Hodge cycles indexed by \mathcal{J} , having level N symplectic similitude structure for any $N \in \mathbb{N}$ and satisfying some additional conditions. This interpretation endows the abelian scheme $\mathcal{A}'_{E(G, X)}$ with a family $(w_{\alpha}^{\mathcal{A}'})_{\alpha \in \mathcal{J}}$ of Hodge cycles (the Betti realizations of pull backs of $w_{\alpha}^{\mathcal{A}'}$ via \mathbb{C} -valued points of $\mathcal{N}_{E(G, X)}$ correspond naturally to v_{α}).

Let $z \in \mathcal{N}'(W(k))$. Let

$$(A, p_A, (\tilde{w}_{\alpha})_{\alpha \in \mathcal{J}}) := z^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'}, (w_{\alpha}^{\mathcal{A}'})_{\alpha \in \mathcal{J}}).$$

Let

$$(M, F^1, \phi, p_M)$$

be the principally quasi-polarized filtered F -crystal over k of the principally quasi-polarized p -divisible group (D, p_D) of (A, p_A) . So p_M is a perfect alternating form on the free $W(k)$ -module M of rank $\dim_{\mathbb{Q}}(W) = 2 \dim_{W(k)}(A)$, F^1 is a maximal isotropic subspace of M with respect to p_M and the pair (M, ϕ) is a Dieudonné module. Let \tilde{t}_α and \tilde{v}_α be the de Rham and respectively the p -component of the étale component of \tilde{w}_α . We have $\tilde{t}_\alpha \in \mathcal{T}(M)[\frac{1}{p}]$. Let \tilde{G} be the Zariski closure in $GL(M)$ of the subgroup of $GL(M[\frac{1}{p}])$ fixing $\tilde{t}_\alpha, \forall \alpha \in \mathcal{J}$.

It is known that \tilde{w}_α is a de Rham cycle; in other words, via de Rham and so also the crystalline Fontaine comparison theory, \tilde{t}_α and \tilde{v}_α correspond to each other. If $A_{B(k)}$ is definable over a number field contained in $B(k)$, then this was known previously (for instance, see [Bl, (0.3)]). The general case follows from loc. cit. and [Va1, 5.2.16] (in [Va1, 5.2] an odd prime is used; however the proof of [Va1, 5.2.16] applies for all primes). In particular we get that $\phi(\tilde{t}_\alpha) = \tilde{t}_\alpha, \forall \alpha \in \mathcal{J}$. Here ϕ acts on $M^*[\frac{1}{p}]$ by mapping $e \in M^*[\frac{1}{p}]$ into $\sigma \circ e \circ \phi^{-1} \in M^*[\frac{1}{p}]$ and so it acts on $\mathcal{T}(M)[\frac{1}{p}]$ in the natural tensor product way.

Let μ be the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512]. It fixes $\tilde{t}_\alpha, \forall \alpha \in \mathcal{J}$ (cf. the functorial aspects of [Wi, p. 513]). So μ factors through \tilde{G} ; we denote also by

$$\mu : \mathbb{G}_{mW(k)} \rightarrow \tilde{G}$$

the resulting factorization. Let

$$M = F^1 \oplus F^0$$

be the direct sum decomposition such that μ fixes F^0 . Warning: often in what follows in connection to different Kodaira–Spencer maps, we will identify naturally $\text{Hom}(F^1, F^0)$ with a direct summand of $\text{End}(M)$ formed by endomorphisms of M annihilating F^0 .

4.2.1. Fact. *Let w be a complex point of $\text{Sh}_H(G, X) = \mathcal{N}_{E(G, X)}$. Let $(A_w, p_{A_w}) := w^*((\mathcal{A}', \mathcal{P}_{\mathcal{A}'})_{E(G, X)})$. We denote by v_α^w the p -component of the étale component of the Hodge cycle $w^*(w_\alpha^{\mathcal{A}'})$ on A_w . Then there are isomorphisms $(H_{\text{ét}}^1(A_w, \mathbb{Z}_p), (v_\alpha^w)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$ taking the perfect bilinear form on $H_{\text{ét}}^1(A_w, \mathbb{Z}_p)$ defined by p_{A_w} into a $\mathbb{G}_{m\mathbb{Z}_p}(\mathbb{Z}_p)$ -multiple of the perfect bilinear form ψ^* on $L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ defined by ψ .*

Proof: Let $w^\infty \in \text{Sh}(G, X)(\mathbb{C})$ lifting w . Let $\mathcal{A}'_{E(G, X)}^\infty$ be the pull back of $\mathcal{A}'_{E(G, X)}$ to $\text{Sh}(G, X)$. We identify canonically $H_{\text{ét}}^1(A_w, \mathbb{Z}_p) = H_{\text{ét}}^1(w^{\infty*}(\mathcal{A}'_{E(G, X)}^\infty), \mathbb{Z}_p)$. From the standard moduli interpretation of $\text{Sh}(G, X)(\mathbb{C})$ applied to $w^\infty \in \text{Sh}(G, X)(\mathbb{C})$ we get that A_w as a complex manifold is $F_w^{0, -1} \backslash W \otimes_{\mathbb{Q}} \mathbb{C} / L_w$, where L_w is a \mathbb{Z} -lattice of W such that $L_w \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ is a $\hat{\mathbb{Z}}$ -lattice of $W \otimes_{\mathbb{Q}} \mathbb{A}_f = W \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ which is $G(\mathbb{A}_f)$ -conjugate to $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and where $W \otimes_{\mathbb{Q}} \mathbb{C} = F_w^{0, -1} \oplus F_w^{-1, 0}$ is the Hodge decomposition of a Hodge \mathbb{Q} -structure on W defined by some element $h_w \in X$ (see [Di1], [Mi2], [Mi3], [Va1, p. 454]). Moreover, the principal polarization p_{A_w} of A_w is defined naturally by a $\mathbb{G}_{m\mathbb{Q}}(\mathbb{Q})$ -multiple of ψ . So $(H_{\text{ét}}^1(A_w, \mathbb{Z}_p), (v_\alpha^w)_{\alpha \in \mathcal{J}})$ is identified naturally with $(L_w^* \otimes_{\mathbb{Z}} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$ and so with a $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ -conjugate of $(L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$. From this and the existence of the mentioned $\mathbb{G}_{m\mathbb{Q}}(\mathbb{Q})$ -multiple of ψ , the Fact follows. This ends the proof.

4.2.2. Theorem. *For the point $z \in \mathcal{N}'(W(k))$ we consider the following condition*

(*) there is a flat, closed subgroup \tilde{G}^1 of \tilde{G} , a set \mathcal{J}^1 containing \mathcal{J} and a family of tensors $(\tilde{t}_\alpha)_{\alpha \in \mathcal{J}^1}$ of $\mathcal{T}(M)[\frac{1}{p}]$ fixed by ϕ and $\mu_{B(k)}$, such that $\tilde{G}_{B(k)}^1$ is the subgroup of $GL(M[\frac{1}{p}])$ fixing \tilde{t}_α , $\forall \alpha \in \mathcal{J}^1$, and for any $W(k)$ -valued point \tilde{g} of the universal smoothening $\tilde{G}^{1'}$ of \tilde{G}^1 lifting a k -valued point of the identity component of $\tilde{G}_k^{1'}$, the F -crystal $(M, \tilde{g}\phi)$ over k has only positive slopes.

If $p = 2$ we assume that either $G_{\mathbb{Z}_{(p)}}$ is a torus or (*) holds. Then we have:

- 1) There is an isomorphism $(M, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}})$.
- 2) The group scheme \tilde{G} is isomorphic to $G_{W(k)}$.

Proof: Part 1) follows from [Va6, 1.2 and 4.3 5)] applied in the context of the pair $(D, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}^1})$.¹ It suffices to prove 2) under the extra assumption that the transcendental degree of k is countable. So there is an $E(G, X)$ -monomorphism $B(k) \hookrightarrow \mathbb{C}$. Let $w \in \mathcal{N}_{E(G, X)}(\mathbb{C})$ be the composite of the resulting morphism $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(B(k))$ with the generic fibre $z_{B(k)}$ of z . From 4.2.1 and 1) we get the existence of an isomorphism $(M, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$. From this 2) follows. This ends the proof.

4.2.3. Remark. We refer to 4.2.2 (*). If $\tilde{G}_{B(k)}^1$ is a reductive subgroup of $\tilde{G}_{B(k)}$ through which $\mu_{B(k)}$ factors and if ϕ normalizes the Lie subalgebra $\text{Lie}(\tilde{G}_{B(k)}^1)$ of $\text{End}(M[\frac{1}{p}]) = M[\frac{1}{p}] \otimes_{B(k)} M^*[\frac{1}{p}]$, then a family of tensors $(\tilde{t}_\alpha)_{\alpha \in \mathcal{J}^1}$ as in 4.2.2 (*) exists (cf. [Va6, 2.5.3]).

4.3. Proof of 1.4. We start the proof of 1.4. So $e(v) = 1$, i.e. the prime v of $E(G, X)$ is unramified over p . Let O be a faithfully flat $O_{(v)}$ -algebra which is a DVR of index of ramification 1. Let Y be a regular, formally smooth O -scheme such that there is a morphism $q_{Y_{E(G, X)}} : Y_{E(G, X)} \rightarrow \text{Sh}_H(G, X) = \mathcal{N}_{E(G, X)}$. So Y is a healthy regular scheme, cf. 1.2.1. So $q_{Y_{E(G, X)}}$ extends uniquely to a morphism $q_Y : Y \rightarrow \mathcal{N}'$, cf. 3.1.1 1). To prove 1.4 1) we just need to show that q_Y factors through \mathcal{N} , cf. 3.1.3, 3.1.5 and def. 1.2 1). It suffices to check this under the extra assumption that $Y = \text{Spec}(R_Y)$, where $R_Y = W(k)[[T_1, \dots, T_m]]$ with $m \in \mathbb{N} \cup \{0\}$ and with T_1, \dots, T_m as independent variables. Let $z_Y \in Y(W(k))$ be defined by the $W(k)$ -epimorphism $R_Y \twoheadrightarrow W(k)$ taking T_i into 0, $i \in \{1, \dots, m\}$. We will use the notations of 4.2 for $z := q_Y \circ z_Y \in \mathcal{N}'(W(k))$. As \mathcal{N} is an open subscheme of \mathcal{N}' (cf. 3.1.5), to show that q_Y factors through \mathcal{N} it suffices to show that z factors through \mathcal{N} . Let \tilde{G}' be the universal smoothening of \tilde{G} , cf. 2.3.

From Fontaine comparison theory we get that $\tilde{G}_{B(k)}$ is isomorphic to $G_{B(k)}$ and so has dimension $l := \dim_{\mathbb{Q}}(G)$. So the relative dimension of \tilde{G}' over $W(k)$ is also l . Let R be the completion of the local ring of \tilde{G}' at the identity element of \tilde{G}'_k . We choose an identification $R = W(k)[[x_1, \dots, x_l]]$ such that the identity section of \tilde{G}' is defined by $x_1 = \dots = x_l = 0$.

¹ The assumption G smooth used in [Va6, 3.4.5 2) and 4.3 5)] is not needed due to the following reason. Let Q , k_0 , \hat{Q}^0 and \tilde{h} be as in [Va6, 3.4.1]. Considering a Teichmüller lift $\text{Spec}(W(k_0)) \rightarrow \text{Spec}(Q)$ factoring through $\text{Spec}(\hat{Q}_0)$, we get that $(M \otimes_{W(k)} W(k_0), h(\phi \otimes \sigma_{k_0}))$ is isomorphic to $(M \otimes_{W(k)} W(k_0), \tilde{h}(\phi \otimes \sigma_{k_0}))$, for some $\tilde{h} \in G^0(W(\bar{k})) \cap G(W(k_0))$. So $(M \otimes_{W(k)} W(k_0), h(\phi \otimes \sigma_{k_0}))$ does not have both slopes 0 and 1 with positive multiplicities.

Let $M_R := M \otimes_{W(k)} R$ and $F_R^1 := F^1 \otimes_{W(k)} R$. Let Φ_R be the Frobenius lift of R compatible with σ and taking x_i into x_i^p . Let $\Phi := g_{\text{univ}}(\phi \otimes \Phi_R)$ be the Φ_R -linear endomorphism of M_R , where $g_{\text{univ}} \in \tilde{G}'(R)$ is the universal element. Let $\Omega_{R/W(k)}^\wedge$ be the p -adic completion of $\Omega_{R/W(k)}$; it is a free R -module having $\{dx_1, \dots, dx_l\}$ as an R -basis. Let $\nabla : M_R \rightarrow M_R \otimes_R \Omega_{R/W(k)}^\wedge$ be the unique connection such that we have $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$, where $d\Phi_R$ is the differential map of Φ_R (cf. [Fa, Th. 10]).

Loc. cit. implies that ∇ is integrable and nilpotent mod p . Moreover we have the following three properties (see [Va6, 3.3 1]) for the first two properties and see [Va6, proof of 5.1] for the third one):

- (i) *the connection ∇ is of the form $\delta + \eta$, where δ is the connection on M_R annihilating $M \otimes 1$ and where $\eta \in (\text{Lie}(\tilde{G}_{B(k)}) \cap \text{End}(M)) \otimes_{W(k)} \Omega_{R/W(k)}^\wedge$;*
- (ii) *the connection on $\mathcal{T}(M_R) = \mathcal{T}(M) \otimes_{W(k)} R$ induced naturally by ∇ annihilates the tensor $\tilde{t}_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$, $\forall \alpha \in \mathcal{J}$;*
- (iii) *the connection ∇ is versal and its Kodaira–Spencer map has as image the direct summand $(\text{Lie}(\tilde{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)) \otimes_{W(k)} R$ of $\text{Hom}(F^1, F^0) \otimes_{W(k)} R$.*

Each element of $\text{Ker}(\mathbb{G}_{mW(k)}(R) \rightarrow \mathbb{G}_{mW(k)}(R/(x_1, \dots, x_l)))$ is of the form $\beta\Phi_R(\beta^{-1})$ with $\beta \in \text{Ker}(\mathbb{G}_{mW(k)}(R) \rightarrow \mathbb{G}_{mW(k)}(R/(x_1, \dots, x_l)))$. So there is a unique $\text{Ker}(\mathbb{G}_{mW(k)}(R) \rightarrow \mathbb{G}_{mW(k)}(R/(x_1, \dots, x_l)))$ -multiple p_{M_R} of the perfect alternating form p_M on M_R such that we have $p_{M_R}(\Phi(x), \Phi(y)) = p\Phi_R(p_{M_R}(x, y))$, $\forall x, y \in M_R$.

The categories of p -divisible groups over $\text{Spec}(R)$ and respectively over $\text{Spf}(R)$ are canonically isomorphic, cf. [dJ1, 2.4.4]; below we will use this fact without any extra comment. Let (D_R, p_{D_R}) be the principally quasi-polarized p -divisible group over $\text{Spec}(R)$ lifting (D, p_D) and whose principally quasi-polarized filtered F -crystal is $(M_R, F_R^1, \Phi, \nabla, p_{M_R})$. The existence up to unique isomorphism of the fibre of (D_R, p_{D_R}) over $\text{Spec}(R/pR)$ is implied by [dJ1, Th. of introd.]. So as the ideal $p(x_1, \dots, x_l)$ of $R/(x_1, \dots, x_l)^m$ has a natural nilpotent PD structure $\forall m \in \mathbb{N}$, the existence up to unique isomorphism of (D_R, p_{D_R}) is implied by Grothendieck–Messing deformation theory.

Let (A_R, p_{A_R}) be the principally polarized abelian scheme over R lifting (A, p_A) and whose principally quasi-polarized p -divisible group is (D_R, p_{D_R}) , cf. Serre–Tate deformation theory and Grothendieck algebraization theorem. Let

$$q_R : \text{Spec}(R) \rightarrow \mathcal{M}$$

be the morphism corresponding to (A_R, p_{A_R}) and the level N symplectic similitude structures of (A_R, p_{A_R}) lifting those of (A, p_A) ($N \in \mathbb{N}$ being prime to p).

Let $d := \dim_{\mathbb{C}}(X)$. Let $y : \text{Spec}(k) \hookrightarrow \mathcal{N}'_{W(k)}$ be the closed embedding defined naturally by the special fibre of z . Let O_y^{big} and O_y be the completions of the local rings of z viewed as $W(k)$ -valued points of $\mathcal{M}_{W(k)}$ and respectively of $\mathcal{N}'_{W(k)}$. The local ring O_y is normal and of dimension $1 + d$, cf. 3.1.1 2) and the fact that any normal $O_{(v)}$ -scheme of finite type is excellent. The natural homomorphism $n_y : O_y^{\text{big}} \rightarrow O_y$ is finite, cf. 3.1.1 3).

The rank r of the direct summand $\text{Lie}(\tilde{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)$ of $\text{Hom}(F^1, F^0)$ is the dimension of the unipotent radical of the maximal parabolic subgroup of $\tilde{G}_{B(k)}$ normalizing

$F^1[\frac{1}{p}]$ and so is the dimension of the compact dual of any connected component of X . So we have $r = d$. From (iii) and the equality $r = d$, we get that there is a natural $W(k)$ -epimorphism $h_R : R \rightarrow \tilde{R} := W(k)[[y_1, \dots, y_d]]$ such that the composite of the $W(k)$ -homomorphism $s_R : O_y^{\text{big}} \rightarrow R$ defining q_R with h_R , is a $W(k)$ -epimorphism $h_y^{\text{big}} : O_y^{\text{big}} \rightarrow \tilde{R}$. In order to show that there is a $W(k)$ -homomorphism $H_y : O_y \rightarrow \tilde{R}$ making the following diagram commutative

$$\begin{array}{ccc} O_y^{\text{big}} & \xrightarrow{n_y} & O_y \\ s_R \downarrow & & \downarrow h_y \\ R & \xrightarrow{h_R} & \tilde{R}, \end{array}$$

we will need to first recall a result of Faltings.

4.3.1. A result of Faltings. As $\tilde{t}_\alpha \in \mathcal{T}(M)[\frac{1}{p}]$ is the de Rham component of the Hodge cycle \tilde{w}_α on $A_{B(k)}$ and due to 4.3 (ii), a result of Faltings implies that $\tilde{t}_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$ is the de Rham component of a Hodge cycle on $A_{R[\frac{1}{p}]}$. As the essence of this result is just outlined in [Va1, 4.1.5], we include a full proof of it here.

As \mathcal{M} is a pro-étale cover of a quasi-projective $\mathbb{Z}_{(p)}$ -scheme (cf. 3.1.1 4)) and as \mathcal{J} is a countable set, it suffices to prove Faltings' result in the case when there is a morphism $e_k : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(W(k))$. We view \mathbb{C} as a $W(k)$ -algebra via e_k . Let $\tilde{R}_{\mathbb{C}} := \mathbb{C}[[y_1, \dots, y_d]]$ and let $R_{\mathbb{C}} := \mathbb{C}[[x_1, \dots, x_l]]$. Let $I_{\mathbb{C}}$ be the maximal ideal of $R_{\mathbb{C}}$.

Let $(A_{R_{\mathbb{C}}}, p_{A_{R_{\mathbb{C}}}}, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}})$ be the pull back of $(A_R, p_{A_R}, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}})$ via the natural $W(k)$ -monomorphism $R = W(k)[[x_1, \dots, x_l]] \hookrightarrow \mathbb{C}[[x_1, \dots, x_l]] = R_{\mathbb{C}}$. It suffices to show that $\tilde{t}_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R_{\mathbb{C}}$ is the de Rham component of a Hodge cycle on $A_{R_{\mathbb{C}}}$.

Let $(B_{\tilde{R}_{\mathbb{C}}}, p_{B_{\tilde{R}_{\mathbb{C}}}}, (w_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}})$ be the pull back of $(\mathcal{A}', \mathcal{P}_{\mathcal{A}'}, (w_{\alpha}^{\mathcal{A}'})_{\alpha \in \mathcal{J}})$ via a formally étale morphism $\text{Spec}(\tilde{R}_{\mathbb{C}}) \rightarrow \mathcal{N}$ whose composite with the natural embedding $\text{Spec}(\mathbb{C}) \hookrightarrow \text{Spec}(\tilde{R}_{\mathbb{C}})$ is $e_k \circ z \in \mathcal{N}'(dbC) = \mathcal{N}(\mathbb{C})$. Let $W_{\tilde{R}_{\mathbb{C}}} := H_{dR}^1(B_{\tilde{R}_{\mathbb{C}}}/\tilde{R}_{\mathbb{C}})$. Let $\psi_{\tilde{R}_{\mathbb{C}}}$ be the perfect alternating form on $W_{\tilde{R}_{\mathbb{C}}}$ defined by $p_{B_{\tilde{R}_{\mathbb{C}}}}$. Let $t_{\alpha}^{\tilde{R}_{\mathbb{C}}} \in \mathcal{T}(W_{\tilde{R}_{\mathbb{C}}})$ be the de Rham component of $w_{\alpha}^{\tilde{R}_{\mathbb{C}}}$. Let ∇_B be the Gauss–Manin connection on $W_{\tilde{R}_{\mathbb{C}}}$ defined by $B_{\tilde{R}_{\mathbb{C}}}$. We denote by ψ^* the alternating form on W^* defined naturally by ψ . It is well known that there are isomorphisms $I_B : (W_{\tilde{R}_{\mathbb{C}}}, \psi_{\tilde{R}_{\mathbb{C}}}, (t_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} \tilde{R}_{\mathbb{C}}, \psi^*, (v_{\alpha})_{\alpha \in \mathcal{J}})$ under which ∇_B becomes the connection on $W^* \otimes_{\mathbb{Q}} \tilde{R}_{\mathbb{C}}$ annihilating $x \otimes 1$, $\forall x \in W^*$ (for instance, see [De3, §2 and §6]). We fix such an isomorphism I_B and we view it as an identification.

By induction on $s \in \mathbb{N}$ we show there is a unique morphism of \mathbb{C} -schemes

$$J_s : \text{Spec}(R_{\mathbb{C}}/I_{\mathbb{C}}^s) \rightarrow \text{Spec}(\tilde{R}_{\mathbb{C}})$$

having the following two properties:

(i) *the kernel of the composite \mathbb{C} -homomorphism $\tilde{R}_{\mathbb{C}} \rightarrow R_{\mathbb{C}}/I_{\mathbb{C}}^s \rightarrow R_{\mathbb{C}}/I_{\mathbb{C}}$ is the ideal (y_1, \dots, y_d) of $\tilde{R}_{\mathbb{C}}$;*

(ii) *there is an isomorphism Q_s between $J_s^*((B_{\tilde{R}_{\mathbb{C}}}, p_{B_{\tilde{R}_{\mathbb{C}}}}, (t_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}}))$ and the reduction mod $I_{\mathbb{C}}^s$ of $(A_{R_{\mathbb{C}}}, p_{A_{R_{\mathbb{C}}}}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}})$ which modulo $I_{\mathbb{C}}/I_{\mathbb{C}}^s$ is defined by $1_{A_{\mathbb{C}}}$.*

As $\mathcal{N}_{E(G,X)}$ is a finite, étale scheme over a closed subscheme of $\mathcal{M}_{E(G,X)}$, the deformation $(B_{\tilde{R}_{\mathbb{C}}}, p_{B_{\tilde{R}_{\mathbb{C}}}})$ of the principally polarized abelian variety $(A, p_A)_{\mathbb{C}}$ is versal and the Kodaira–Spencer map of ∇_B is injective having as image a free $\tilde{R}_{\mathbb{C}}$ -module of rank d . This implies the uniqueness of J_s . The existence of J_1 is obvious. The passage from the existence of J_s to the existence of J_{s+1} goes as follows. Let $J'_{s+1} : \text{Spec}(R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}) \rightarrow \text{Spec}(\tilde{R}_{\mathbb{C}})$ be an arbitrary morphism of \mathbb{C} -schemes lifting J_s . Let $\nabla_{B,s+1}$ be the connection on $W_{\tilde{R}_{\mathbb{C}}} \otimes_{\tilde{R}_{\mathbb{C}}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1} = W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}$ which is the extension of ∇_B via J'_{s+1} (the identification being defined by I_B).

We recall that for any $u \in \mathbb{N}$ and for every abelian scheme $\pi_C : C \rightarrow Z$ over a smooth \mathbb{C} -scheme Z , we have a natural isomorphism of complex sheaves

$$(1) \quad R^u \pi_{C^{\text{an}*}}(\mathbb{C}) \xrightarrow{\sim} R^u \pi_{C^{\text{an}*}}(\Omega_{C^{\text{an}}/Z^{\text{an}}})^{\nabla_C^{\text{an}}}$$

on the complex manifold Z^{an} . Here $\pi_{C^{\text{an}*}} : C^{\text{an}} \rightarrow Z^{\text{an}}$ (resp. ∇_C^{an}) is the morphism of complex manifolds associated naturally to π_C (resp. is the connection on $R^u \pi_{C^{\text{an}*}}(\Omega_{C^{\text{an}}/Z^{\text{an}}})$ induced by the Gauss–Manin connection on $R^u \pi_{C*}(\Omega_{C/Z})$).

For any $\beta \in \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$, there are isomorphisms of $(W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}, (v_{\alpha})_{\alpha \in \mathcal{J}})$ taking ψ^* into $\beta\psi^*$. So based on the proof of 4.2.1 and the construction of M_R we get that there are isomorphisms

$$I_A : (M_R \otimes_R R_{\mathbb{C}}, p_{M_R}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}, \psi^*, (v_{\alpha})_{\alpha \in \mathcal{J}}).$$

We consider a deformation \tilde{A}_Z of $A_{R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}}$ over a smooth \mathbb{C} -scheme Z . Let \tilde{A}_Z^t be the dual abelian scheme of \tilde{A}_Z . By applying (1) for $u = 1$ and for $C = \tilde{A}_Z$ and based on 4.3 (ii), we get that we can choose I_A such that under it the Gauss–Manin connection $\nabla_{A,s+1}$ on $M_R \otimes_R R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}$ becomes the connection on $W^* \otimes_{\mathbb{C}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}$ annihilating $x \otimes 1$, $\forall x \in W^*$. We will view the reduction $I_{A,s+1}$ of $I_A \bmod I_{\mathbb{C}}^{s+1}$ as an identification.

We fix an isomorphism D_{s+1} between $J'_{s+1}((W_{\tilde{R}_{\mathbb{C}}}, \psi_{\tilde{R}_{\mathbb{C}}}, (t_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}})) = (W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}, \psi^*, (v_{\alpha})_{\alpha \in \mathcal{J}})$ and $(M_R \otimes_R R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}, p_{M_R}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}}) = (W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}, \psi^*, (v_{\alpha})_{\alpha \in \mathcal{J}})$ with the properties that:

(iii) modulo $I_{\mathbb{C}}^s$ it is the isomorphism defined by Q_s ;

(iv) it respects the Gauss–Manin connections, i.e. it takes $\nabla_{B,s+1}$ into $\nabla_{A,s+1}$.

Let $F_{A,s+1}^1$ and $F_{B,s+1}^1$ be the Hodge filtrations of $W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}$ defined naturally by $A_{R_{\mathbb{C}}}$ and respectively by $B_{\tilde{R}_{\mathbb{C}}}$. The direct summands $F_{A,s+1}^1$ and $D_{s+1}(F_{B,s+1}^1)$ of $W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}$ coincide modulo $I_{\mathbb{C}}^s/I_{\mathbb{C}}^{s+1}$, cf. (iii). Moreover, there are direct sum decompositions $W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1} = F_{A,s+1}^1 \oplus F_{A,s+1}^0 = F_{B,s+1}^1 \oplus F_{B,s+1}^0$ defined naturally by cocharacters $\mu_{A,s+1}$ and respectively $\mu_{B,s+1}$ of the subgroup $G_{R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}}$ of $GL(W^* \otimes_{\mathbb{Q}} R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1})$. Argument: the existence of $\mu_{A,s+1}$ is a direct consequence of the existence of the cocharacter $\mu : \mathbb{G}_{mW(k)} \rightarrow \tilde{G}$ (see paragraph before 4.2.1) and of the definition of F_R^1 (see 4.3) while the existence of $\mu_{B,s+1}$ is well known. As $F_{A,s+1}^1$ and $D_{s+1}(F_{B,s+1}^1)$ coincide modulo $I_{\mathbb{C}}^s/I_{\mathbb{C}}^{s+1}$ (cf. (iii)), we can choose $\mu_{A,s+1}$ and $\mu_{B,s+1}$ such that $D_{s+1}^{-1}\mu_{A,s+1}D_{s+1}$ and $\mu_{B,s+1}$ coincide modulo $I_{\mathbb{C}}^s/I_{\mathbb{C}}^{s+1}$. So based on [SGA3, Vol. II, p. 47–48], there is

$g_{s+1} \in \text{Ker}(G(R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}) \rightarrow G(R_{\mathbb{C}}/I_{\mathbb{C}}^s))$ such that $D_{s+1}^{-1}\mu_{A,s+1}D_{s+1} = g_{s+1}\mu_{B,s+1}g_{s+1}^{-1}$. So $D_{s+1}(g_{s+1}(F_{B,s+1}^1)) = F_{A,s+1}^1$.

As the Kodaira–Spencer map of ∇_B is injective and its image is a free $\tilde{R}_{\mathbb{C}}$ -module of rank d and as $\nabla_{B,s+1}(v_{\alpha}) = 0, \forall \alpha \in \mathcal{J}$, we get that we can replace J'_{s+1} by another morphism $J_{s+1} : \text{Spec}(R_{\mathbb{C}}/I_{\mathbb{C}}^{s+1}) \rightarrow \text{Spec}(\tilde{R}_{\mathbb{C}})$ lifting J_s and such that under it and I_B the Hodge filtration $F_{B,s+1}^1$ gets replaced by $g_{s+1}(F_{B,s+1}^1)$. So D_{s+1} becomes the de Rham realization of an isomorphism Q_{s+1} between $J_{s+1}^*((B_{\tilde{R}_{\mathbb{C}}}, p_{B_{\tilde{R}_{\mathbb{C}}}}, (t_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}}))$ and the reduction mod $I_{\mathbb{C}}^{s+1}$ of $(A_{R_{\mathbb{C}}}, p_{A_{R_{\mathbb{C}}}}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}})$ and which lifts Q_s . So J_{s+1} has the desired properties. This ends the induction.

Let $J_{\infty} : \text{Spec}(R_{\mathbb{C}}) \rightarrow \text{Spec}(\tilde{R}_{\mathbb{C}})$ be the morphism defined by J_s 's ($s \in \mathbb{N}$). The isomorphism Q_s is uniquely determined by (i) and (ii) and so Q_{s+1} lifts Q_s . So we get the existence of an isomorphism

$$Q_{\infty} : J_{\infty}^*((B_{\tilde{R}_{\mathbb{C}}}, p_{B_{\tilde{R}_{\mathbb{C}}}}, (t_{\alpha}^{\tilde{R}_{\mathbb{C}}})_{\alpha \in \mathcal{J}})) \xrightarrow{\sim} (A_{R_{\mathbb{C}}}, p_{A_{R_{\mathbb{C}}}}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}})$$

which modulo $I_{\mathbb{C}}$ is defined by $1_{A_{\mathbb{C}}}$. So $\tilde{t}_{\alpha} \in \mathcal{T}(M) \otimes_{W(k)} R_{\mathbb{C}}$ is the de Rham component of a Hodge cycle on $A_{R_{\mathbb{C}}}$, $\forall \alpha \in \mathcal{J}$. This ends the argument for Faltings' result.

4.3.2. End of the proof of 1.4. The existence of Q_{∞} implies that the $W(k)$ -epimorphism $h_y^{\text{big}} : O_y^{\text{big}} \rightarrow \tilde{R}$ (see paragraph before 4.3.1) factors through O_y . By reasons of dimensions, the resulting $W(k)$ -epimorphism $h_y : O_y \rightarrow \tilde{R}$ is an isomorphism. So $\mathcal{N}'_{W(k)}$ is formally smooth at z and so z factors through \mathcal{N} . So we can also view y as a k -valued point of $\mathcal{N}_{W(k)}$. So the $O_{(v)}$ -scheme \mathcal{N} together with the natural action of $G(\mathbb{A}_f^{(p)})$ on it is the weak integral canonical model of $\text{Sh}_H(G, X)$ over $O_{(v)}$. So 1.4 1) holds. As h_y is an isomorphism, the natural $W(k)$ -homomorphism $O_y^{\text{big}} \rightarrow O_y$ is also an isomorphism. So the natural $W(k)$ -morphism $\mathcal{N}_{W(k)} \rightarrow \mathcal{M}_{W(k)}$ is a formally closed embedding at the k -valued point y of $\mathcal{N}_{W(k)}$. As the morphism q_Y of the beginning of 4.3 was arbitrary, the role of z is that of an arbitrary $W(k)$ -valued of \mathcal{N}' (and so based on 1.4 1)) of \mathcal{N} . So the $W(k)$ -morphism $\mathcal{N}_{W(k)} \rightarrow \mathcal{M}_{W(k)}$ is a formally closed embedding at any k -valued point of $\mathcal{N}_{W(k)}$. So 1.4 2) also holds. This ends the proof of 1.4.

4.3.3. Simple properties. We denote also by q_R its factorization through \mathcal{N} or \mathcal{N}' . As h_y is an isomorphism and as we have an epimorphism $h_R : R \rightarrow \tilde{R}$, the morphism $q_R : \text{Spec}(R) \rightarrow \mathcal{N}$ is formally smooth. We have a canonical identification $H_{dR}^1(q_R^*(\mathcal{A}')/R) = M_R$ and under it the perfect form on M_R defined by the principal polarization $q_R^*(\mathcal{P}_{\mathcal{A}'})$ of $q_R^*(\mathcal{A}')$ is p_{M_R} , cf. the definition of $q_R : \text{Spec}(R) \rightarrow \mathcal{M}$.

Also the pull back of $w_{\alpha}^{\mathcal{A}'}$ via $q_{R[\frac{1}{p}]}$ is a Hodge cycle on $A_{R[\frac{1}{p}]}$ having $\tilde{t}_{\alpha} \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$ as its de Rham component. This follows either from the existence of Q_{∞} or from the fact that there is no non-trivial tensor of $\mathcal{T}(M) \otimes_{W(k)} (x_1, \dots, x_l)[\frac{1}{p}]$ fixed by Φ .

4.4. Lemma. *We recall that we use the notations of 1.3. We assume that one of the following two conditions holds:*

1 The original approach of Faltings was using the strictness of filtrations of morphisms between Hodge \mathbb{R} -structures to show the existence of the element g_{s+1} .

(i) *there is a smooth, affine group scheme $G_{\mathbb{Z}_{(p)}}^0$ over $\mathbb{Z}_{(p)}$ extending G , whose special fibre $G_{\mathbb{F}_p}^0$ has the same rank as G and such that there is a homomorphism $G_{\mathbb{Z}_{(p)}}^0 \rightarrow G_{\mathbb{Z}_{(p)}}$ extending the identity automorphism of G ;*

(ii) *the group scheme $G_{\mathbb{Z}_{(p)}}$ is quasi-reductive for (G, X, v) in the sense of 1.2 3).*

Then $e(v) = 1$ and the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty.

Proof: We first assume (i) holds. Each torus of $G_{\mathbb{F}_p}^0$ lifts to a torus of $G_{\mathbb{Z}_p}^0$ (cf. [SGA3, Vol. II, p. 47–48]) and so $G_{\mathbb{Z}_p}^0$ has tori of rank equal to the rank of G . Let $T_{\mathbb{Z}_{(p)}}^0$ be a torus of $G_{\mathbb{Z}_{(p)}}^0$ of the same rank as G and such that there is $h \in X$ factoring through $T_{\mathbb{R}}^0$. Its existence is implied by [Ha, Lemma 5.5.3]. The pair $(T_{\mathbb{Q}}^0, \{h\})$ is a Shimura pair. Each prime of $E(T_{\mathbb{Q}}^0, \{h\})$ dividing v is unramified over p (cf. [Mi3, 4.6 and 4.7]) and so we have $e(v) = 1$. The intersection $H^0 := H \cap T_{\mathbb{Z}_{(p)}}^0(\mathbb{Q}_p)$ is the unique hyperspecial subgroup $T_{\mathbb{Z}_{(p)}}^0(\mathbb{Z}_p)$ of $T_{\mathbb{Z}_{(p)}}^0(\mathbb{Q}_p)$. So the integral canonical model \mathcal{T}^0 of $\mathrm{Sh}_{H^0}(T_{\mathbb{Q}}^0, \{h\})$ over $E(T_{\mathbb{Q}}^0, \{h\})_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(v)}$ exists and has non-trivial fibres of dimension 0 (cf. [Va1, 3.2.8]). So \mathcal{T}^0 as an $\mathcal{O}_{(v)}$ -scheme is formally étale. So as \mathcal{N} has the smooth extension property (cf. 1.4 1)) and as \mathcal{T}^0 is formally étale over $\mathcal{O}_{(v)}$, the functorial morphism $\mathrm{Sh}_{H^0}(T_{\mathbb{Q}}^0, \{h\}) \rightarrow \mathrm{Sh}_H(G, X)$ of $E(G, X)$ -schemes extends uniquely to a morphism $\mathcal{T}^0 \rightarrow \mathcal{N}$ of $\mathcal{O}_{(v)}$ -schemes. So the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty as $\mathcal{T}_{k(v)}^0$ is so.

We now assume (ii) holds. Let $G_{\mathbb{Z}_p}^1$ and μ_v be as in 1.2 3). Let $T_{\mathbb{F}_p}^1$ be a maximal torus of $G_{\mathbb{F}_p}^1$. Due to the existence of μ_v , $T_{\mathbb{F}_p}^1$ has positive rank. The torus $T_{\mathbb{F}_p}^1$ lifts to a torus $T_{\mathbb{Z}_p}^1$ of $G_{\mathbb{Z}_p}^1$, cf. [SGA3, Vol. II, p. 47–48]. Let $T_{0\mathbb{Q}_p}^0$ be a maximal torus of $G_{\mathbb{Q}_p}$ having $T_{\mathbb{Q}_p}^1$ as a subtorus. Let T^0 be a maximal torus of G such that there is $h \in X$ factoring through $T_{\mathbb{R}}^0$ and such that $T_{\mathbb{Q}_p}^0$ is $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ -conjugate to $T_{0\mathbb{Q}_p}^0$. Again, its existence is implied by [Ha, Lemma 5.5.3]. Not to introduce extra notations, we will assume $T_{0\mathbb{Q}_p}^0 = T_{\mathbb{Q}_p}^0$.

The intersection $H^0 := H \cap T^0(\mathbb{Q}_p)$ is not necessarily the maximal compact, open subgroup of $T^0(\mathbb{Q}_p)$ and the subgroup $T^0(\mathbb{Q})H^0$ of $T^0(\mathbb{Q}_p)$ is not necessarily $T^0(\mathbb{Q}_p)$ itself. However, the intersection $T_{\mathbb{Z}_p}^1(\mathbb{Q}_p) \cap H$ is the unique hyperspecial subgroup $T_{\mathbb{Z}_p}^1(\mathbb{Z}_p)$ of $T_{\mathbb{Q}_p}^1(\mathbb{Q}_p)$. We fix an $\mathcal{O}_{(v)}$ -monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ as in 1.2 3). As μ_h and μ_v are $G(\mathbb{C})$ -conjugate and as $G_{\mathbb{C}}^1$ is a normal subgroup of $G_{\mathbb{C}}$, μ_h factors through the intersection $T_{\mathbb{C}}^1 = T_{\mathbb{C}}^0 \cap G_{\mathbb{C}}^1$. So as $T_{\mathbb{Z}_p}^1$ splits over $W(\mathbb{F})$, we get that the field of definition $E(T_{\mathbb{Q}}^0, \{h\})$ of μ_h is a number subfield of \mathbb{C} containing $E(G, X)$ and such that all primes of it dividing v are unramified over p . So $e(v) = 1$. From class field theory (see [Lan, Th. 4 of p. 220]) and the reciprocity map of [Mi2, p. 163–164] we easily get that each connected component of $\mathrm{Sh}_{H^0}(T_{\mathbb{Q}}^0, \{h\})_{\mathbb{C}}$ is the spectrum of an abelian extension of $E(T_{\mathbb{Q}}^0, \{h\})$ unramified over any prime of $E(T_{\mathbb{Q}}^0, \{h\})$ dividing v . So as in [Va1, 3.2.8] we argue that there is an integral canonical model \mathcal{T}^0 of $\mathrm{Sh}_{H^0}(T_{\mathbb{Q}}^0, \{h\})$ over $E(T_{\mathbb{Q}}^0, \{h\})_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{(v)}$. As above, this implies that the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty. This ends the proof.

4.5. An application. We assume $G_{\mathbb{Q}_p}$ is unramified. Let \tilde{H} be a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. We show that we can modify the \mathbb{Z} -lattice L of W and if needed f such that we have $H = \tilde{H}$. Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the reductive group scheme over $\mathbb{Z}_{(p)}$ having G as its generic

fibre and having \tilde{H} as its group of \mathbb{Z}_p -valued points (see beginning of §3). Let $\tilde{L}_{(p)}$ be a $\mathbb{Z}_{(p)}$ -lattice of W such that the monomorphism $G \hookrightarrow GL(W)$ extends to a homomorphism $\tilde{G}_{\mathbb{Z}_{(p)}} \rightarrow GL(\tilde{L}_{(p)})$, cf. [Ja, 10.9 of Part I]. Let \tilde{L} be the \mathbb{Z} -lattice of W such that we have $\tilde{L}[\frac{1}{p}] = L[\frac{1}{p}]$ and $\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \tilde{L}_{(p)}$. If ψ induces a perfect form on \tilde{L} , then \tilde{L} is the searched for \mathbb{Z} -lattice. Argument: as \tilde{H} is a maximal compact subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, the monomorphism $\tilde{H} \hookrightarrow G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap GL(\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p)(\mathbb{Z}_p)$ is an isomorphism. If ψ does not induces a perfect form on \tilde{L} , then we modify f as follows.

Let $L'_1 := \tilde{L} \oplus \tilde{L}^*$. Let $W_1 := L'_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. Let ψ'_1 be a perfect alternating form on L'_1 such that the group scheme $SL(\tilde{L})$, when viewed naturally as a subgroup of $SL(L'_1)$, is in fact a subgroup of $Sp(L'_1, \psi'_1)$. So \tilde{L} and \tilde{L}^* are both maximal isotropic \mathbb{Z} -lattices of W_1 with respect to ψ'_1 . Let $\tilde{G}_{\mathbb{Z}_{(p)}}^0$ be the Zariski closure in $\tilde{G}_{\mathbb{Z}_{(p)}}$ of the identity component of the subgroup of G fixing ψ (one can check that the subgroup of G fixing ψ is in fact connected). It is a reductive subgroup of $SL(\tilde{L})$ and so also of $GSp(L'_1, \psi'_1)$. The subgroup of $GSp(L'_1, \psi'_1)$ generated by $Z(GL(L'_1))$ and $\tilde{G}_{\mathbb{Z}_{(p)}}^0$ is a reductive group scheme (cf. 2.2.5 2)) and it is naturally identified with $\tilde{G}_{\mathbb{Z}_{(p)}}$. We fix an $h \in X$. The Hodge structure on W_1 defined by h is of type $\{(-1, 0), (0, -1)\}$. So as in the proof of 3.3 we get that there is a perfect alternating form ψ_1 on W_1 normalized by G , inducing a perfect form on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and such $2\pi i \psi_1$ is a polarization of the Hodge \mathbb{Q} -structure on W_1 defined by h . We get an injective map $f_1 : (G, X) \hookrightarrow (GSp(W_1, \psi_1), S_1)$. Let L_1 be a \mathbb{Z} -lattice of W_1 such that ψ_1 induces a perfect alternating form on L_1 and $L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. As above we argue that $\tilde{H} = G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap GL(L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p)(\mathbb{Z}_p)$. As $\tilde{G}_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, the condition 4.4 (i) holds in the context of (f_1, L_1, v) . So from 1.4 1) and 4.4 we get:

4.5.1. Corollary. *Let (G, X) be a Shimura pair of Hodge type. Let v a prime of the reflex field $E(G, X)$ dividing a prime p with the property that the group $G_{\mathbb{Q}_p}$ is unramified. Then for any hyperspecial subgroup \tilde{H} of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, there is a weak integral canonical model \mathcal{N} of $\text{Sh}_{\tilde{H}}(G, X)$ over $O_{(v)}$.*

This Corollary can be viewed as a complete solution to Langlands conjecture of [La, p. 411] for Shimura varieties of Hodge type.

4.5.2. A more general form of 4.5.1. We recall that we use the notations of 1.3. Let \tilde{H} be a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $\tilde{G}_{\mathbb{Z}_p}$ be a smooth, affine group scheme over \mathbb{Z}_p having $G_{\mathbb{Q}_p}$ as its generic fibre and such that \tilde{H} is its group of \mathbb{Z}_p -valued points (cf. [Ti2, p. 52]). Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the smooth, affine group scheme over $\mathbb{Z}_{(p)}$ having G as its generic fibre and whose extension to \mathbb{Z}_p is $\tilde{G}_{\mathbb{Z}_p}$, cf. [Va1, 3.1.3.1]. As above we argue that there is an injective map $f_1 : (G, X) \hookrightarrow (GSp(W_1, \psi_1), S_1)$ and a \mathbb{Z} -lattice L_1 of W_1 such that ψ_1 induces a perfect alternating form on L_1 and the monomorphism $f_1 : G \hookrightarrow GSp(W_1, \psi_1)$ extends to a homomorphism $\tilde{G}_{\mathbb{Z}_{(p)}} \rightarrow GSp(L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi_1)$ which is not necessarily a closed embedding. The maximal property of \tilde{H} implies $\tilde{H} = G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap GSp(L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p, \psi_1)(\mathbb{Z}_p)$. So if the special fibre $\tilde{G}_{\mathbb{F}_p}$ of $\tilde{G}_{\mathbb{Z}_p}$ has a torus of the same rank as G , then condition 4.4 (i) holds and so from 1.4 1) and 4.4 we again get that $\text{Sh}_{\tilde{H}}(G, X)$ has a weak integral canonical model over $O_{(v)}$.

4.6. A complement to 4.3. We use the notations of 4.3 used in the proof of 1.4. Let

$$\mathcal{C}_{\text{univ}} := (M_R, F_R^1, \Phi, \nabla, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, p_{M_R}).$$

We recall that $M_R := M \otimes_{W(k)} R$, that $F_R^1 := F^1 \otimes_{W(k)} R$ and that $\Phi := g_{\text{univ}}(\phi \otimes \Phi_R)$, where $g_{\text{univ}} \in \tilde{G}'(R)$ is the universal element. We also recall that there is a principally quasi-polarized p -divisible group (D, p_D) over $W(k)$ whose principally quasi-polarized filtered F -crystal over k is (M, F^1, ϕ, p_M) (see 4.2), that $R_Y = W(k)[[T_1, \dots, T_m]]$, that $Y = \text{Spec}(R_Y)$ and that $z_Y \in Y(W(k))$ is defined by the ideal (T_1, \dots, T_m) of R_Y . Let Φ_{R_Y} be the Frobenius lift of R_Y compatible with σ and taking T_i into T_i^p , $\forall i \in \{1, \dots, m\}$.

If \tilde{F}^1 is a direct summand of a free R_Y -module \tilde{M} of finite rank, then we endow $\mathcal{T}(\tilde{M})$ with the tensor product filtration $(F^i(\mathcal{T}(\tilde{M})))_{i \in \mathbb{Z}}$ defined by the filtrations $(F^i(\tilde{M}))_{i \in \{0, 1\}}$ and $(F^i(\tilde{M}^*))_{i \in \{-1, 0\}}$ of \tilde{M} and respectively \tilde{M}^* , where $F^1(\tilde{M}) := \tilde{F}^1$, $F^0(\tilde{M}) := \tilde{M}$, $F^{-1}(\tilde{M}^*) := \tilde{M}^*$ and $F^0(\tilde{M}^*) := \{x \in \tilde{M}^* | x(\tilde{F}^1) = \{0\}\}$. If $\tilde{\Phi}$ is a Φ_{R_Y} -linear monomorphism of \tilde{M} , then we let $\tilde{\Phi}$ act in the natural tensor way on $\mathcal{T}(\tilde{M})[\frac{1}{p}]$; so if $e \in \tilde{M}^*$, then $\tilde{\Phi}(e) \in \tilde{M}^*[\frac{1}{p}]$ is such that we have $\tilde{\Phi}(e)(\tilde{\Phi}(x)) = \Phi_{R_Y}(e(x)) \in R_Y$, $\forall x \in \tilde{M}$.

Let $(\tilde{M}, \tilde{F}^1, \tilde{\Phi}, \tilde{\nabla}, (\tilde{t}_\alpha^Y)_{\alpha \in \mathcal{J}}, p_{\tilde{M}})$ be a principally quasi-polarized filtered F -crystal over $\text{Spec}(R_Y/pR_Y)$ endowed with a family of tensors $(\tilde{t}_\alpha^Y)_{\alpha \in \mathcal{J}}$ of $\mathcal{T}(\tilde{M})[\frac{1}{p}]$ such that the following three axioms hold:

- (i) $\tilde{\Phi}$ induces an R_Y -linear isomorphism $(\tilde{M} + \frac{1}{p}\tilde{F}^1) \otimes_{R_Y} \Phi_{R_Y} R_Y \xrightarrow{\sim} \tilde{M}$;
- (ii) each \tilde{t}_α^Y is fixed by $\tilde{\Phi}$, is annihilated by $\tilde{\nabla}$ and belongs to $F^0(\mathcal{T}(\tilde{M}))[\frac{1}{p}]$;
- (iii) modulo $I_Y := (T_1, \dots, T_m)$ it is $(M, F^1, \phi, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, p_M)$.

4.6.1. Theorem. *There is a morphism $i_Y : Y \rightarrow \text{Spec}(R)$ of $W(k)$ -schemes such that $g_{\text{univ}} \circ i_Y \circ z_Y$ is the identity section of \tilde{G}' and $i_Y^*(\mathcal{C}_R)$ is isomorphic to \mathcal{C}_Y under an isomorphism which modulo I_Y becomes the identity automorphism of 1_M .*

Proof: If \tilde{G} is smooth, then the Theorem is implied by [Fa, Th. 10 and rm. iii) after it]. We now treat the general case following the proof of [Va6, 5.1]. Let (D_Y, p_{D_Y}) be the unique principally quasi-polarized p -divisible group over Y lifting (D, p_D) and whose principally quasi-polarized filtered F -crystal is $(\tilde{M}, \tilde{F}^1, \tilde{\Phi}, \tilde{\nabla}, p_{\tilde{M}})$; its existence and uniqueness is argued in the same way we argued them for (D_R, p_{D_R}) of 4.3.

By induction on $s \in \mathbb{N}$ we show that there is a morphism $i_{Y,s} : \text{Spec}(R_Y/I_Y^s) \rightarrow \text{Spec}(R)$ of $W(k)$ -schemes such that $i_{Y,s}^*((D_R, p_{D_R}))$ is isomorphic to (D_Y, p_{D_Y}) modulo I_Y^s under an isomorphism DIS_s having the following two properties:

- (i) it lifts the identity automorphism of (D, p_D) ;
- (ii) it defines an isomorphism IS_s between $i_{Y,s}^*(\mathcal{C}_R)$ and \mathcal{C}_Y modulo I_Y^s which modulo I_Y is the identity automorphism of 1_M .

As $\Phi_Y(T_i) = T_i^p$, such an isomorphism IS_s is unique. The uniqueness of DIS_1 is obvious. If $s = 1$ we take $i_{Y,s}$ to be defined by the $W(k)$ -epimorphism $R \twoheadrightarrow R/(x_1, \dots, x_l) = W(k) = R_Y/I_Y$ and we take DIS_1 and IS_1 to be defined by the identity automorphism of (D, p_D) and respectively by 1_M .

The passage from s to $s + 1$ goes as follows. We endow the ideal $J_{Y,s} := I_Y^s/I_Y^{s+1}$ of R_Y/I_Y^{s+1} with the trivial PD structure; so $J_{Y,s}^{[2]} = \{0\}$. Based on Grothendieck–Messing deformation theory, the uniqueness of DIS_{s+1} is implied by the uniqueness of DIS_s and of IS_{s+1} . So to end the induction we are left to show that IS_{s+1} and DIS_{s+1} exist.

Let $\tilde{i}_{Y,s+1} : \text{Spec}(R_Y/I_Y^{s+1}) \rightarrow \text{Spec}(R)$ be an arbitrary morphism of $W(k)$ -schemes through which $i_{Y,s}$ factors naturally. We write

$$\tilde{i}_{Y,s+1}^*(M_R, F_R, \Phi, \nabla, p_{M_R}) = (M \otimes_{W(k)} R_Y/I_Y^{s+1}, F^1 \otimes_{W(k)} R_Y/I_Y^{s+1}, \Phi_{s+1}, \nabla_{s+1}, p_{M_R}).$$

Due to existence of DIS_s , from Grothendieck–Messing deformation theory we get that there is a direct summand of $M \otimes_{W(k)} R_Y/I_Y^{s+1}$ lifting $F^1 \otimes_{W(k)} R_Y/I_Y^s$ and such that the quadruple $(\tilde{M}, \tilde{F}, \tilde{\Phi}, \tilde{\nabla}, p_{\tilde{M}})$ modulo I_Y^{s+1} is isomorphic to the quadruple $(M \otimes_{W(k)} R_Y/I_Y^{s+1}, F_{s+1}^1, \Phi_{s+1}, \nabla_{s+1}, p_{M_R})$ under an isomorphism $\tilde{I}S_{s+1}$ lifting the one defined by IS_s . As $\Phi_{s+1}(M \otimes_{W(k)} J_{Y,s}) = \{0\}$ and as $\tilde{I}S_{s+1}$ lifts IS_s , under $\tilde{I}S_{s+1}$ the tensor \tilde{t}_α^Y modulo I_Y^{s+1} becomes \tilde{t}_α .

The rest of the inductive argument is entirely as in the last four paragraphs of [Va6, proof of 5.1]. Briefly, let U^{big} and U be the maximal unipotent, closed subgroups of $GL(M)$ and respectively of \tilde{G} on which μ acts via the identity character of $\mathbb{G}_{mW(k)}$ (see loc. cit.). Let $u_{s+1} \in \text{Lie}(U^{\text{big}}) \otimes_{W(k)} J_{Y,s}$ be the unique element such that we have $(1_{M \otimes_{W(k)} R_Y/I_Y^{s+1}} + u_{s+1})(F^1 \otimes_{W(k)} R_Y/I_Y^{s+1}) = F_{s+1}^1$. As in loc. cit. we argue that $u_{s+1} \in \text{Lie}(U) \otimes_{W(k)} J_{Y,s}$. The image of the Kodaira–Spencer map of ∇ is the tensorization with R of the direct summand $\text{Lie}(U)$ of $\text{Hom}(F^1, F^0) = \text{Lie}(U^{\text{big}})$, cf. 4.3 (iii). So as in loc. cit. we can replace $\tilde{i}_{Y,s+1}$ by another morphism $i_{Y,s+1} : \text{Spec}(R_Y/I_Y^{s+1}) \rightarrow \text{Spec}(R)$ through which $i_{Y,s}$ still factors and for which F_{s+1}^1 gets replaced by (i.e. becomes) $F^1 \otimes_{W(k)} R_Y/I_Y^{s+1}$. So from Grothendieck–Messing deformation theory we get that $i_{Y,s+1}^*((D_R, p_{D_R}))$ is isomorphic to (D_Y, p_{D_Y}) modulo I_Y^{s+1} under an isomorphism DIS_{s+1} which lifts DIS_s and which defines an isomorphism IS_{s+1} between $i_{Y,s+1}^*(\mathcal{C}_R)$ and \mathcal{C}_Y modulo I_Y^{s+1} . As DIS_{s+1} lifts DIS_s , the uniqueness of I_s implies that IS_{s+1} lifts IS_s and so also IS_1 . This ends the induction.

We take $i_Y : Y \rightarrow \text{Spec}(R)$ such that it lifts $i_{Y,s}$, $\forall s \in \mathbb{N}$. From the very definition of $i_{Y,1}$ we get that $g_{\text{univ}} \circ i_Y \circ z_Y$ is the identity section of \tilde{G}' . Moreover, $i_Y^*(\mathcal{C}_R)$ is isomorphic to \mathcal{C}_Y under an isomorphism lifting IS_s , $\forall s \in \mathbb{N}$. This ends the proof.

§5. Proof of 1.5

We recall from 4.2 that k is an algebraically closed field of characteristic p and that the notations L , $L_{(p)}$, $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$, $E(G, X)$, v , $k(v)$, $O_{(v)}$, $K_p = GSp(L, \psi)(\mathbb{Z}_p)$, $G_{\mathbb{Z}_{(p)}}$, $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, \mathcal{M} , \mathcal{N}' , \mathcal{N} , $(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ are as in 1.3. For simplicity we will also assume that k is of countable transcendental degree. In the next two Chapters, π_{G_0} , $(v_\alpha)_{\alpha \in \mathcal{J}}$, $(w_\alpha^{A'})_{\alpha \in \mathcal{J}}$ will be as in 4.2 and to a point $z \in \mathcal{N}'(W(k))$ we will associate $(A, p_A, (\tilde{w}_\alpha)_{\alpha \in \mathcal{J}})$, $(M, F^1, \phi, p_M, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}})$, $M = F^1 \oplus F^0$ and $\mu : \mathbb{G}_{mW(k)} \rightarrow \tilde{G}$ as in 4.2. In this Chapter we first prove 1.5 (see 5.1 to 5.4) and then at the end of it we include two refinements of this proof needed in §6 (see 5.5 and 5.6).

Let $R_0 := W(k)[[x]]$, where x is an independent variable. Let Φ_{R_0} be the Frobenius lift of R_0 compatible with σ and taking x into x^p .

5.1. Basic notations and facts. We start proving 1.5 by introducing some basic notations and facts. We have $e(v) = 1$. We recall that \mathcal{N} is an open subscheme of \mathcal{N}' , cf. 3.1.5. So $\mathcal{N}_{k(v)}$ is also an open subscheme of $\mathcal{N}'_{k(v)}$. Moreover, the open embedding $\mathcal{N} \hookrightarrow \mathcal{N}'$ is a pro-étale cover of an open embedding between $O_{(v)}$ -schemes of finite type (cf. proof of 3.1.2) and the $k(v)$ -scheme $\mathcal{N}_{k(v)}$ is non-empty (cf. 4.4). So to show that $\mathcal{N}_{k(v)}$ is a non-empty, open closed subscheme of $\mathcal{N}'_{k(v)}$ we just need to show that for any commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[[x]]) & \longleftarrow & \mathrm{Spec}(k((x))) \\ \downarrow y & & \downarrow q & & \downarrow q_{k((x))} \\ \mathcal{N}' & \longleftarrow & \mathcal{N}'_{k(v)} & \longleftarrow & \mathcal{N}_{k(v)}, \end{array}$$

the morphism $y : \mathrm{Spec}(k) \rightarrow \mathcal{N}'$ factors through the subscheme $\mathcal{N}_{k(v)}$ of \mathcal{N}' .

We consider the principally quasi-polarized filtered F -crystal

$$(M_0, \Phi_0, \nabla_0, p_{M_0})$$

over $k[[x]]$ of $q^*((\mathcal{A}', \mathcal{P}_{\mathcal{A}'}) \times_{\mathcal{N}'} \mathcal{N}'_{k(v)})$. So M_0 is a free R_0 -module of rank equal to $\dim_{\mathbb{Q}}(W)$, Φ_0 is a Φ_{R_0} -linear endomorphism of M_0 and ∇_0 is an integrable and nilpotent mod p connection on M_0 such that we have $\nabla_0 \circ \Phi_0 = (\Phi_0 \otimes d\Phi_{R_0}) \circ \nabla_0$.

Let O be the local ring of R_0 which is a DVR of mixed characteristic. Let \hat{O} be the completion of O . Let $k_1 := k((x))$. We fix a Teichmüller lift

$$\mathrm{Spec}(W(k_1)) \rightarrow \mathrm{Spec}(R_0)$$

with respect to Φ_{R_0} ; under it $W(k_1)$ gets naturally the structure of an O -algebra and so also of an \hat{O} -algebra. Let $\Phi_{\hat{O}}$ be the Frobenius lift of \hat{O} defined by Φ_{R_0} via localization and completion.

As \mathcal{N} is formally smooth over $O_{(v)}$, there is a lift $\tilde{z}_1 : \mathrm{Spec}(\hat{O}) \rightarrow \mathcal{N}$ of the morphism $\mathrm{Spec}(k((x))) \rightarrow \mathcal{N}$ defined naturally by q . Let

$$(\tilde{A}_1, p_{\tilde{A}_1}, (\tilde{w}_{1\alpha})_{\alpha \in \mathcal{J}}) := \tilde{z}_1^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'}, (\tilde{w}_{\alpha})_{\alpha \in \mathcal{J}}).$$

Let $\tilde{t}_{1\alpha}$ be the de Rham realization of $\tilde{w}_{1\alpha}$. We identify canonically $M_0 \otimes_{R_0} \hat{O} = H_{dR}^1(\tilde{A}_1/\hat{O})$ (cf. [Be, 2.3 of Ch. V]) and so we view $\tilde{t}_{1\alpha}$ as a tensor of $\mathcal{T}(M_0 \otimes_{R_0} \hat{O})[\frac{1}{p}]$.

For $\alpha \in \mathcal{J}$ let $n(\alpha) \in \mathbb{N} \cup \{0\}$ be such that $v_{\alpha} \in W^{*\otimes n(\alpha)} \otimes_{\mathbb{Q}} W^{\otimes n(\alpha)} \subset \mathcal{T}(W^*)$. We get the existence of $n_{\alpha} \in \mathbb{N} \cup \{0\}$ such that we have

$$p^{n_{\alpha}} \tilde{t}_{1\alpha} \in (M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n_{\alpha}}) \otimes_{R_0} \hat{O} \subset \mathcal{T}(M_0 \otimes_{R_0} \hat{O}).$$

5.1.1. Proposition. *We have $p^{n_{\alpha}} \tilde{t}_{1\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n(\alpha)} \subset \mathcal{T}(M_0)$, $\forall \alpha \in \mathcal{J}$.*

Proof: The tensor $p^{n_{\alpha}} \tilde{t}_{1\alpha}$ is fixed by the natural σ_{k_1} -linear endomorphism of $\mathcal{T}(M_0 \otimes_{R_0} B(k_1))$ defined by Φ_0 (see 4.2). So $p^{n_{\alpha}} \tilde{t}_{1\alpha}$ is also fixed by the natural $\Phi_{\hat{O}}$ -linear endomorphism of $\mathcal{T}(M_0 \otimes_{R_0} \hat{O})[\frac{1}{p}]$ defined by Φ_0 .

The field $k((x))$ has $\{x\}$ as a p -basis, i.e. $\{1, x, \dots, x^{p-1}\}$ is a basis of $k((x))$ over $k((x))^p = k((x^p))$. So the p -adic completion of the \hat{O} -module $\Omega_{\hat{O}/W(k)}$ of relative differentials is naturally isomorphic to $\hat{O}dx$, cf. [BM, 1.3.1].

The de Rham component of $w_{\alpha}^{A'}$ is annihilated by the Gauss–Manin connection of \mathcal{A}' (this is a property of Hodge cycles, for instance it follows from [De3, 2.5] applied in the context of a quotient of $\mathrm{Sh}_H(G, X)$ by a small compact, open subgroup of $G(\mathbb{A}_f^{(p)})$). So the tensor $p^{n_{\alpha}}\tilde{t}_{1\alpha}$ is annihilated by the Gauss–Manin connection on $\mathcal{T}(H_{dR}^1(\tilde{A}_1/\hat{O})) = \mathcal{T}(M_0 \otimes_{R_0} \hat{O})$ of \tilde{A}_1 and so also by the p -adic completion of this last connection. In other words, $p^{n_{\alpha}}\tilde{t}_{1\alpha}$ is annihilated by the connection $\nabla_0 : M_0 \otimes_{R_0} \hat{O} \rightarrow M_0 \otimes_{R_0} \hat{O}dx$ which is the natural extension of the connection ∇_0 on M_0 and so denoted also by ∇_0 .

As the field $k((x))$ has a p -basis, any F -crystal over $k((x))$ is uniquely determined by its evaluation at the thickening naturally associated to the closed embedding $\mathrm{Spec}(k((x))) \hookrightarrow \mathrm{Spec}(\hat{O})$ (cf. [BM, 1.3.3]). So the natural identification $(M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n_{\alpha}}) \otimes_{R_0} \hat{O} = \mathrm{End}(M_0^{\otimes n(\alpha)} \otimes_{R_0} \hat{O})$ allows us to view naturally $p^{n_{\alpha}}\tilde{t}_{1\alpha}$ as an endomorphism of the F -crystal over $k((x))$ defined by the tensor product of $n(\alpha)$ -copies of $(M_0 \otimes_{R_0} \hat{O}, \Phi_0 \otimes \Phi_{\hat{O}}, \nabla_0)$. So from 4.1 we get that $p^{n_{\alpha}}\tilde{t}_{1\alpha}$ can be viewed naturally as an endomorphism of the F -crystal over $k[[x]]$ defined by the tensor product of $n(\alpha)$ -copies of (M_0, Φ_0, ∇_0) . So we have $p^{n_{\alpha}}\tilde{t}_{1\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n_{\alpha}} \subset \mathcal{T}(M_0)$. This ends the proof of the Proposition.

5.1.2. Some group schemes. To state the next Proposition we need several notations pertaining to group schemes. Let $G_{\mathbb{Z}_p}^1$ be a reductive subgroup of $G_{\mathbb{Z}_p}$ as in 1.2 3). Warning: it is not necessarily the pull back to $\mathrm{Spec}(\mathbb{Z}_p)$ of a closed subgroup of $G_{\mathbb{Z}_{(p)}}$. However, by enlarging $(v_{\alpha})_{\alpha \in \mathcal{J}}$, we can assume that the tensor $\pi_{G_{\mathbb{Q}_p}^1}$ of 4.2 is a \mathbb{Q}_p -linear combination of v_{α} 's. Let $\tilde{\pi}^1$ be the corresponding \mathbb{Q}_p -linear combination of $\tilde{t}_{1\alpha}$'s. Let $n^1 \in \mathbb{N} \cup \{0\}$ be such that $p^{n^1}\tilde{\pi}^1 \in \mathrm{End}(\mathrm{End}(M_0))$, cf. 5.1.1. We consider the product decomposition

$$G_{\mathbb{Q}_p}^{1\mathrm{ad}} = \prod_{i \in I_p} G_{\mathbb{Q}_p}^{1\mathrm{iad}}$$

in \mathbb{Q}_p -simple, adjoint groups. Let $G_{\mathbb{Q}_p}^{1\mathrm{ider}}$ be the normal subgroup of $G_{\mathbb{Q}_p}^{1\mathrm{der}}$ having $G_{\mathbb{Q}_p}^{1\mathrm{iad}}$ as its adjoint. Similarly we get that there is $n^{1i} \in \mathbb{N} \cup \{0\}$ such that $p^{n^{1i}}\tilde{\pi}^{1i} \in \mathrm{End}(\mathrm{End}(M_0))$; here $\tilde{\pi}^{1i}$ corresponds to $\pi_{G_{\mathbb{Q}_p}^{1\mathrm{ider}}}$ in the same way $\tilde{\pi}^1$ corresponded to $\pi_{G_{\mathbb{Q}_p}^1}$.

By enlarging $(v_{\alpha})_{\alpha \in \mathcal{J}}$, we can assume that each element of $\mathrm{End}(L_{(p)})$ fixed by $G_{\mathbb{Z}_{(p)}}$ is v_{α_0} for some $\alpha_0 \in \mathcal{J}$. Let $Z_{\mathbb{Z}_p}^1$ be the center of the centralizer of $Z^0(G_{\mathbb{Z}_p}^1)$ in $GL(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$. Let $AL(Z_{\mathbb{Z}_p}^1)$ be the semisimple \mathbb{Z}_p -subalgebra of $\mathrm{End}(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ whose elements are the elements of $\mathrm{Lie}(Z_{\mathbb{Z}_p}^1)$. Each $e \in AL(Z_{\mathbb{Z}_p}^1)$ is a \mathbb{Z}_p -linear combination of endomorphisms of $L_{(p)}$ fixed by $G_{\mathbb{Z}_{(p)}}$ and so is identified naturally with a \mathbb{Z}_p -endomorphism e of \mathcal{A}' . For simplicity we denote also by $e \in \mathrm{End}(M_0)$ the crystalline realization of the \mathbb{Z}_p -endomorphism $q^*(e)$ of $q^*(\mathcal{A}' \times_{\mathcal{N}'} \mathcal{N}'_{k(v)})$.

Let η_0 be the field of fractions of R_0 . Let $\tilde{G}_{0\eta_0}$ be the subgroup of $GL(M_0)_{\eta_0}$ fixing $p^{n_{\alpha}}\tilde{t}_{1\alpha}$, $\forall \alpha \in \mathcal{J}$; this makes sense due to 5.1.1. Let \tilde{G}_0 be the Zariski closure of $\tilde{G}_{0\eta_0}$ in $GL(M_0)$. Let $\tilde{G}_{0\eta_0}^1$ be the normal, connected subgroup of $\tilde{G}_{0\eta_0}$ whose Lie algebra is

$\tilde{\pi}^1(\text{Lie}(\tilde{G}_{0\eta_0}))$. The uniqueness of $\tilde{G}_{0\eta_0}^1$ is implied by [Bo, 7.1 of Ch. 1]. So to check the existence of $\tilde{G}_{0\eta_0}^1$ and to show that $\tilde{G}_{0\eta_0}^1$ is reductive, it suffices to show that $\tilde{G}_{0B(k_1)}^1$ exists and is reductive. But $\tilde{G}_{0B(k_1)}^1$ corresponds via Fontaine comparison theory for $A_{1B(k_1)}$ to $G_{\mathbb{Q}_p}^1$ and so it exists and is reductive. Similarly we argue that $\tilde{G}_{0\eta_0}$ is a reductive group. Let

$$\tilde{G}_0^1$$

be the Zariski closure of $\tilde{G}_{0\eta_0}^1$ in $GL(M_0)$.

5.2. Key Proposition. *The closed subscheme \tilde{G}_0^1 of $GL(M_0)$ is a reductive subgroup.*

Proof: We first show that the Zariski closure $Z^0(\tilde{G}_0^1)$ of $Z^0(\tilde{G}_{0\eta_0}^1)$ in $GL(M_0)$ is a subtorus of $GL(M_0)$. As R_0 is strictly henselian, the semisimple R_0 -subalgebra $AL(Z_{\mathbb{Z}_p}^1) \otimes_{\mathbb{Z}_p} R_0$ of $\text{End}(M_0)$ is a product of copies of R_0 and so is the Lie algebra of a split subtorus $\tilde{Z}_{R_0}^1$ of $GL(M_0)$. But $Z^0(\tilde{G}_{0\eta_0}^1)$ is naturally a subtorus of $\tilde{Z}_{\eta_0}^1$ and so $Z^0(\tilde{G}_0^1)$ is a subtorus of $\tilde{Z}_{R_0}^1$ and so also of $GL(M_0)$. Based on 2.2.5, to prove the Proposition we just need to show that the Zariski closure $\tilde{G}_0^{1\text{der}}$ of $\tilde{G}_{0\eta_0}^{1\text{der}}$ in $GL(M_0)$ is a semisimple group scheme. Let V be an arbitrary local ring of R_0 which is a DVR. Based on 2.2.3 and 2.2.4, to prove the last statement, it suffices to show that $\tilde{G}_{1V}^{0\text{der}}$ is a semisimple subgroup of $GL(M_0)_V$. We have to consider two Cases: $V = O$ and V is of equal characteristic 0.

Case 1. Let V be O . Let $z_1 \in \mathcal{N}(W(k_1)) \subset \mathcal{N}'(W(k_1))$ be the composite of $\tilde{z}_1 \in \mathcal{N}(\hat{O})$ with the natural morphism $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(\hat{O})$. Let $(A_1, p_{A_1}) := z_1^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'}) = (\tilde{A}_1, p_{\tilde{A}_1})_{W(k_1)}$. Let $(M_1, F_1^1, \phi_1, p_{M_1})$ be the principally quasi-polarized filtered F -crystal over k_1 of (A_1, p_{A_1}) . Let $\tilde{G}_1, \mu_1 : \mathbb{G}_{mW(k_1)} \rightarrow \tilde{G}_1, R_1$ and

$$\mathcal{C}_{1\text{univ}} = (M_{1R_1}, F_{1R_1}^1, \Phi_1, \nabla_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}, p_{M_{1R_1}})$$

be the analogues of $\tilde{G}, \mu : \mathbb{G}_{mW(k)} \rightarrow \tilde{G}, R$ and respectively $\mathcal{C}_{\text{univ}}$ of 4.2, 4.3 and 4.6 but obtained working with $z_1 \in \mathcal{N}(W(k_1))$ instead of some $z \in \mathcal{N}'(W(k))$. We can identify naturally $M_1 = M_0 \otimes_{R_0} W(k_1)$ and so we also view each tensor $\tilde{t}_{1\alpha}$ as a tensor of $\mathcal{T}(M_1)[\frac{1}{p}]$ and we view the group scheme $\tilde{G}_{0W(k_1)}^1$ as a normal, flat, closed subgroup of the flat, closed subgroup $\tilde{G}_1 = \tilde{G}_{0W(k_1)}$ of $GL(M_1)$. We consider also the morphism

$$q_{R_1} : \text{Spec}(R_1) \rightarrow \mathcal{M}$$

which is the analogue of q_R of 4.3 but obtained working with $z_1 \in \mathcal{N}(W(k_1))$; it factor through \mathcal{N} (see 4.3.2 for the case of q_R).

We first show that if $p = 2$, then condition 4.2.2 (*) holds for the quadruple $(M_1, F_1^1, \phi_1, \tilde{G}_1)$ and for the flat, closed subgroup $\tilde{G}_{0W(k_1)}^1$ of $\tilde{G}_1 = \tilde{G}_{0W(k_1)}$.

We fix an arbitrary $O_{(v)}$ -monomorphism $W(k_1) \hookrightarrow \mathbb{C}$ and an isomorphism $\rho_{1\mathbb{C}} : (M_1 \otimes_{W(k_1)} \mathbb{C}, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} \mathbb{C}, (v_{\alpha})_{\alpha \in \mathcal{J}})$ such that $F_1^1 \otimes_{W(k_1)} \mathbb{C}$ is mapped into the Hodge filtration of $W^* \otimes_{\mathbb{Q}} \mathbb{C}$ defined by a cocharacter $\mu_h : \mathbb{G}_{m\mathbb{C}} \rightarrow G_{\mathbb{C}}$ introduced in 1.1 (see proof of 4.2.1; see also [Va1, 4.1]). From 1.2 3) we get that each such μ_h factors through

$G_{0\mathbb{C}}^1$. So there is a cocharacter $\mu_0^{1'}$ of $\tilde{G}_{0\mathbb{C}}^1$ which is $\tilde{G}_{0\mathbb{C}}^1(\mathbb{C})$ -conjugate to any $\rho_{1\mathbb{C}}^{-1}\mu_h\rho_{1\mathbb{C}}$ and which acts via the inverse of the identical cocharacter of $\mathbb{G}_{m\mathbb{C}}$ on $F_1^1 \otimes_{W(k_1)} \mathbb{C}$ and trivially on $M_1 \otimes_{W(k_1)} \mathbb{C}/F_1^1 \otimes_{W(k_1)} \mathbb{C}$. It is easy to see that $\mu_{1\mathbb{C}}$ and $\mu_0^{1'}$ are $\tilde{G}_0(\mathbb{C})$ -conjugate (to be compared with [Va1, 5.3.1]). So μ_1 factors through $\tilde{G}_{0W(k_1)}^1$.

Let $\tilde{G}_{0W(k_1)}^{1'}$ and \tilde{G}'_1 be the universal smoothenings of $\tilde{G}_{0W(k_1)}^1$ and respectively \tilde{G}_1 . We have a natural homomorphism $\tilde{G}_{0W(k_1)}^{1'} \rightarrow \tilde{G}'_1$, cf. the universal property of \tilde{G}'_1 . Under this homomorphism, the identity component \tilde{C}_1^1 of $\tilde{G}_{0k_1}^{1'}$ maps into the identity component \tilde{C}_1 of \tilde{G}'_{1k_1} . As $\tilde{\pi}^1$ is fixed by ϕ_1 , $\text{Lie}(\tilde{G}_{B(k_1)}^1)$ is normalized by ϕ . Moreover $\tilde{G}_{B(k_1)}^1$ is a reductive group and μ_1 factors through $\tilde{G}_{0W(k_1)}^1$. So from 4.2.3 we get that 4.2.2 (*) holds for $(M_1, F_1^1, \phi_1, \tilde{G}_1)$ and for the smooth subgroup $\tilde{G}_{0W(k_1)}^1$ of \tilde{G}_1 iff we have:

(i) *all slopes of $(M_1, \tilde{g}_1\phi_1)$ are positive, for any element $\tilde{g}_1 \in \tilde{G}_{0W(k_1)}^{1'}(W(k_1))$ which mod p belongs to $\tilde{C}_1^1(k_1)$.*

We show that the assumption that (i) does not hold leads to a contradiction. This assumption implies that there is an open, Zariski dense subscheme \tilde{U}_1 of \tilde{C}_1 such that for any $\tilde{g}_0 \in \tilde{G}_0(W(k_1))$ lifting a k_1 -valued point of \tilde{U}_1 , the F -crystal $(M_1, \tilde{g}_0\phi_1)$ has slope 0 with positive multiplicity. So the pull backs of $\mathcal{C}_{1\text{univ}}$ via geometric points of R_1/pR_1 mapping into \tilde{U}_1 have slope 0 with positive multiplicity. The morphism $q_{R_1} : \text{Spec}(R_1) \rightarrow \mathcal{M}$ factors through \mathcal{N} and it is associated naturally to $\mathcal{C}_{1\text{uni}}$ (and the special fibre of z_1). We conclude that there are geometric points of $\mathcal{N}_{k(v)}$ such that the p -ranks of the corresponding pull backs of \mathcal{A}' are positive. But this contradicts the extra hypothesis of 1.5 for $p = 2$. So for $p = 2$, the condition 4.2.2 (*) holds in the context of $(M_1, F_1^1, \phi_1, \tilde{G}_1)$.

So we can apply 4.2.2 2) to $z_1 \in \mathcal{N}(W(k_1))$ even for $p = 2$. So from the proof of 4.2.2 2) we get that regardless who p is, there is an isomorphism $\rho_1 : (M_1, (t_{1\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k_1), (v_\alpha)_{\alpha \in \mathcal{J}})$. Such an isomorphism is unique up to its (left) composite with an element of $G_{\mathbb{Z}_{(p)}}^1(W(k))$. So as $G_{\mathbb{Z}_p}^1$ is a normal subgroup of $G_{\mathbb{Z}_p}$ and as $\tilde{G}_{0B(k_1)}^1$ corresponds to $G_{\mathbb{Q}_p}^1$ via Fontaine comparison theory for $A_{1B(k_1)}$, we get that the isomorphism $GL(M_1) \xrightarrow{\sim} GL(L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k_1))$ induced naturally by ρ_1 takes $\tilde{G}_{0W(k_1)}^1$ into $G_{W(k_1)}^1$. So $\tilde{G}_{0W(k_1)}^1$ is a reductive group scheme. So $\tilde{G}_{0V}^{1\text{der}}$ is a semisimple group scheme.

Case 2. Let now V be of equal characteristic 0. Let L_V^i be the image of $\text{End}(M_0 \otimes_{R_0} V)$ through $p^{n^{1i}} \tilde{\pi}^{1i}$. Let $L_V := \bigoplus_{i \in I_p} L_V^i$. The $\tilde{\pi}^{1i}$'s are commuting projectors of $\text{End}(M_0[\frac{1}{p}])$ whose images are pairwise intersecting just in $\{0\}$. So L_V as a V -module is a direct summand of $\text{End}(M_0 \otimes_{R_0} V)$ which when tensored with η_0 is $\text{Lie}(\tilde{G}_{0\eta_0}^{1\text{der}})$. So L_V is a Lie subalgebra of $\text{End}(M_0 \otimes_{R_0} V)$. As $\tilde{\pi}^{1i}$ is a projector of $\text{End}(M_0 \otimes_{R_0} V)$, the restriction TR_V^i to L_V^i of the trace form on $\text{End}(M_0 \otimes_{R_0} V)$ is perfect. The Killing form on L_V^i is a non-zero rational multiple of TR_V^i as this is so after tensorization with η_0 . So the Killing form on L_V^i is perfect. So also the Killing form on L_V is perfect. So the tensorizations of L_V with V -algebras which are fields are semisimple (cf. [Hu, 5.1]) and so any derivation of them is inner (cf. [Hu, 5.3]).

Let $\text{Aut}(L_V)$ be the group scheme of Lie automorphisms of L_V . It is an affine group scheme over V whose Lie algebra is L_V (due tho the fact that all mentioned derivations

are inner). So the identity components of the fibres of $\text{Aut}(L_V)$ are semisimple groups. We easily get that the identity component $\text{Aut}^0(L_V)$ of $\text{Aut}(L_V)$ is an adjoint group scheme whose generic fibre is $\tilde{G}_{0\eta_0}^{\text{lad}}$ (argument: an inner automorphism can not specialize to an outer automorphism). So the normalization of $\text{Aut}^0(L_V)$ in $\tilde{G}_{0\eta_0}^{\text{1der}}$ is a semisimple group scheme $\tilde{G}_{0V}^{\text{2der}}$ over V extending $\tilde{G}_{0\eta_0}^{\text{1der}}$. To the Lie monomorphism $L_V \hookrightarrow \text{End}(M_0 \otimes_{R_0} V)$ corresponds a homomorphism $h : \tilde{G}_{0V}^{\text{2der}} \rightarrow GL(M_0)_V$ extending the monomorphism $\tilde{G}_{0\eta_0}^{\text{1der}} = \tilde{G}_{0\eta_0}^{\text{2der}} \rightarrow GL(M_0)_{\eta_0}$ (this is so as the DVR V is of equal characteristic 0). As $\text{Ker}(h_{\eta_0})$ is trivial, from 2.2.2 we get that h is a closed embedding. So $\tilde{G}_{0V}^{\text{1der}} = \tilde{G}_{0V}^{\text{2der}}$ is a semisimple group scheme. This ends the argument for the Proposition.

5.3. Applications of 5.2. Let F_0^1/pF_0^1 be the kernel of $\Phi_0 \bmod p$; it is a free module over $k[[x]] = R_0/pR_0$ of rank equal to half the rank of M_0 . As μ_1 factors through $\tilde{G}_{0W(k_1)}^1$, the subgroup of $\tilde{G}_{0k_1}^1$ normalizing $F_0^1/pF_0^1 \otimes_{k[[x]]} k_1$ is parabolic and so it is the natural pull back of a parabolic subgroup $\tilde{P}_{0k((x))}^1$ of $\tilde{G}_{0k((x))}^1$. The $k[[x]]$ -scheme of parabolic subgroups of $\tilde{G}_{0k[[x]]}^1$ is a projective $k[[x]]$ -scheme, cf. [SGA3, Vol. III, Cor. 3.5 of p. 445]. So the Zariski closure $\tilde{P}_{0k[[x]]}^1$ of $\tilde{P}_{0k((x))}^1$ in $\tilde{G}_{0k[[x]]}^1$ is a parabolic subgroup of $\tilde{G}_{0k[[x]]}^1$. So as \tilde{G}_0^1 is a split reductive group scheme and as μ_{1k_1} factors through $\tilde{G}_{0k_1}^1$, there is a cocharacter $\mu_{0k[[x]]} : \mathbb{G}_{mk[[x]]} \rightarrow \tilde{G}_{0k[[x]]}^1$ factoring through $\tilde{P}_{0k[[x]]}^1$ and producing a direct sum decomposition $M_0/pM_0 = F_0^1/pF_0^1 \oplus F_0^0/pF_0^0$ (with $\beta \in \mathbb{G}_{mk[[x]]}(k[[x]])$ acting through $\mu_{0k[[x]]}$ on F_0^i/pF_0^i via the multiplication with β^{-i} , $i \in \{0, 1\}$). We choose a cocharacter

$$\mu_0 : \mathbb{G}_{mR_0} \rightarrow \tilde{G}_0^1$$

lifting $\mu_{0k[[x]]}$, cf. [SGA3, Vol. II, p. 47–48]. Let $M_0 = F_0^1 \oplus F_0^0$ be the direct sum decomposition such that $\beta \in \mathbb{G}_{mR_0}(R_0)$ acts through μ_0 on F_0^i via the multiplication with β^{-i} , $i \in \{0, 1\}$; the notations match, i.e. the reduction mod p of F_0^i is as defined above.

Let

$$(M, F^1, \phi, \tilde{G}, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, p_M) := (M_0, F_0^1, \Phi_0, \nabla_0, \tilde{G}_0, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}, p_{M_0}) \otimes_{R_0} W(k),$$

where the $W(k)$ -epimorphism $R_0 \twoheadrightarrow W(k)$ has (x) as its kernel. If $p = 2$, then all slopes of (M_1, ϕ_1) are positive (cf. the extra hypothesis of 1.5 for $p = 2$) and so via specialization we get that all slopes of (M, ϕ) are positive. So either $p > 2$ or $p = 2$ and moreover all slopes of (M, ϕ) are positive. So there is a unique p -divisible group D over $W(k)$ whose filtered F -crystal over k is (M, F^1, ϕ) , cf. [Va6, 2.2.3]. To p_M corresponds a principal quasi-polarization p_D of D . So the sextuple

$$\mathcal{C}_0 := (M_0, F_0^1, \Phi_0, \nabla_0, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}, p_{M_0})$$

is a principally quasi-polarized filtered F -crystal over $k[[x]]$ endowed with a family of crystalline tensors for which the axioms 4.6 (i) to (iii) hold.

5.4. End of the proof of 1.5. Let

$$(A_k, p_{A_k}) := y^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'}).$$

As in 4.3, starting from (D, p_D) and $(M_0, F_0^1, \Phi_0, \nabla_0, p_{M_0})$ we construct a principally quasi-polarized p -divisible group $(D_{R_0}, p_{D_{R_0}})$ over R_0 and a homomorphism

$$q_{R_0} : \text{Spec}(R_0) \rightarrow \mathcal{M}$$

lifting the composite of y with the morphism $\mathcal{N} \rightarrow \mathcal{M}$ and such that the principally quasi-polarized p -divisible group of $q_{R_0}^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ is $(D_{R_0}, p_{D_{R_0}})$. We consider the composite

$$z_2 : \text{Spec}(W(k_1)) \rightarrow \mathcal{M}$$

of the dominant morphism $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_0)$ of 5.1 with q_{R_0} . We will next check that q_{R_0} factors through \mathcal{N}' . For this, it suffices to check that z_2 factors through \mathcal{N}' .

Let (A_2, p_{A_2}) be the pull back via z_2 of the universal principally polarized abelian scheme over \mathcal{M} . The principally quasi-polarized filtered F -crystal of (A_2, p_{A_2}) is canonically identified with $(M_1, F_2^1, \phi_1, p_{M_1})$, where F_2^1 is a direct summand of M_1 of rank equal to half the rank of M_1 . Let $(F_2^i(\mathcal{T}(M_1)))_{i \in \mathbb{Z}}$ be the tensor product filtration of $\mathcal{T}(M_1)$ defined by F_2^1 (see 4.6). For $\alpha \in \mathcal{J}$ the tensor $\tilde{t}_{1\alpha} \in \mathcal{T}(M_0)[\frac{1}{p}]$ is annihilated by ∇_0 , is fixed by Φ_0 and belongs to $F_0^0(\mathcal{T}(M_0))[\frac{1}{p}]$, where $(F_0^i(\mathcal{T}(M_0)))_{i \in \mathbb{Z}}$ is the tensor product filtration of $\mathcal{T}(M_0)$ defined by F_0^1 . This implies that $\tilde{t}_{1\alpha} \in F_2^0(\mathcal{T}(M_1))[\frac{1}{p}]$, $\forall \alpha \in \mathcal{J}$. So as before 4.2.1 we argue that the canonical split cocharacter of (M_1, F_2^1, ϕ_1) defined in [Wi, p. 512] factors through $\tilde{G}_{0W(k_1)}$; let $\mu_2 : \mathbb{G}_{mW(k_1)} \rightarrow \tilde{G}_{0W(k_1)}$ be the resulting factorization. As for μ_1 we argue that we can also view μ_2 as a cocharacter $\mu_2 : \mathbb{G}_m \rightarrow \tilde{G}_{0W(k_1)}^1$.

If $G_{\mathbb{Z}_p}^1 = G_{\mathbb{Z}_p}$, then $\tilde{G}_0 = \tilde{G}_0^1$ is a reductive group scheme over R_0 (cf. 5.2) and so \tilde{G} itself is a reductive group scheme over $W(k)$. So the fact that z_2 factors through \mathcal{N}' is proved in [Va5, 6.4.1 i) and ii)]. If $G_{\mathbb{Z}_p}^1 \neq G_{\mathbb{Z}_p}$, then the proof of loc. cit. applies entirely provided we work with the triple $(M_1, \phi_1, \tilde{G}_{0W(k_1)}^1)$ which in [Va5] is called a Shimura F -crystal over k_1 . This is so due to the following two reasons. First μ_1 and μ_2 are conjugate under an element of $\tilde{G}_{0W(k_1)}^1(W(k_1))$ which mod p normalizes the kernel $F_1^1/pF_1^1 = F_2^1/pF_2^1$ of ϕ_1 mod p , cf. [Va5, 3.1.2]. Second the fact that the image of the Kodaira–Spencer map of the connection ∇_1 (of Case 1 of 5.2) can be identified with the maximal direct summand of $\text{Lie}(\tilde{G}_{0W(k_1)})$ and so also of $\text{Lie}(\tilde{G}_{0W(k_1)}^1)$ on which μ_1 acts via the identity character of $\mathbb{G}_{mW(k_1)}$ (see 4.3 (iii) for the case of ∇), applies entirely as in [Va5, proof of 6.4.1] to show that there is a morphism $z_3 : \text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_1)$ such that the pull back of $\mathcal{C}_{1\text{univ}}$ through it is $(M_1, F_2^1, \phi_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}, p_{M_1})$. We denote also by $z_3 \in \mathcal{M}(W(k_1))$ the composite of $z_3 : \text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_1)$ with $q_{R_1} : \text{Spec}(R_1) \rightarrow \mathcal{M}$ (of Case 1 of 5.2); it factors through \mathcal{N} (as q_{R_1} does) and so we also view z_3 as a $W(k_1)$ -valued point of \mathcal{N} or \mathcal{N}' . As for $p = 2$ the special fibre of A_1 has p -rank 0, for p arbitrary we get that the principally quasi-polarized p -divisible groups of the pull backs of $(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ via z_2 and z_3 are the same (cf. [Va6, 2.2.3]). So from Serre–Tate deformation theory we get that z_2 and z_3 are the same $W(k_1)$ -valued points of \mathcal{M} and so also of \mathcal{N}' .

So z_2 factors through \mathcal{N}' . So q_{R_0} factors through \mathcal{N}' and so (cf. 1.4 1)) also through \mathcal{N} . So the morphisms $q : \text{Spec}(k[[x]]) \rightarrow \mathcal{N}'_{k(v)}$ and $y : \text{Spec}(k) \rightarrow \mathcal{N}'$ factor through $\mathcal{N}_{k(v)}$. This ends the argument that $\mathcal{N}_{k(v)}$ is a non-empty, open closed subscheme of $\mathcal{N}'_{k(v)}$ and so ends the proof of 1.5.

5.5. Lemma. *We use the above notations of 5.1.2, of Case 1 of 5.2 and of 5.3. We do not assume that the hypotheses of 1.5 hold but we just assume that $e(v) = 1$ and that \tilde{G}_0 is a reductive subgroup of $GL(M_0)$. Then there is an isomorphism*

$$\rho_{1z} : (M_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}, p_{M_1}) \xrightarrow{\sim} (M \otimes_{W(k)} W(k_1), (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, p_M).$$

Proof: Let $\mathcal{C}_{\text{univ}}$ be constructed as in 4.6 starting from $(M, F^1, \phi, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, p_M)$. Its construction makes sense even if we do not know if the composite z of $q_{R_0} : \text{Spec}(R_0) \rightarrow \mathcal{M}$ with the natural $W(k)$ -epimorphism $\text{Spec}(W(k)) \hookrightarrow \text{Spec}(R_0)$ defined by the locus of $x = 0$, factors or not through \mathcal{N}' . From 4.6.1 we get that \mathcal{C}_0 of 5.3 is the pull back of $\mathcal{C}_{\text{univ}}$ via a morphism $i_{\text{Spec}(R_0)} : \text{Spec}(R_0) \rightarrow \text{Spec}(R) = \text{Spec}(W(k)[[x_1, \dots, x_l]])$ of $W(k)$ -schemes which at the level of $W(k)$ -algebras maps the ideal (x_1, \dots, x_l) of R into the ideal (x) of R_0 . The existence of $i_{\text{Spec}(R_0)}$ implies the existence of ρ_{1z} . This ends the proof.

5.6. A refinement. In this section we assume $p = 2$. The extra hypothesis of 1.5 that the pull backs of \mathcal{A}' via geometric points of $\mathcal{N}_{k(v)}$ have all slopes positive is too restrictive; it has been used in Case 1 of 5.2 as well as in 5.4. So we now explain how we can considerably weaken it under some extra hypotheses. Let $\tilde{v}_{1\alpha}$ be the p -component of the étale component of the Hodge cycle $\tilde{w}_{1\alpha}$ on $\tilde{A}_{1B(k_1)} = A_{1B(k_1)}$ (cf. notations before 5.1.1). We list three additional conditions:

- (i) *the group scheme $G_{\mathbb{Z}_{(p)}}$ is reductive and we have $G_{\mathbb{Z}_p}^1 = G_{\mathbb{Z}_p}$;*
- (ii) *there is an isomorphism $(M_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H_{\text{ét}}^1(A_{1B(k_1)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k_1), (\tilde{v}_{1\alpha})_{\alpha \in \mathcal{J}})$;*
- (iii) *the quadruple (M, F^1, ϕ, p_M) is the principally quasi-polarized filtered F -crystal of a principally polarized abelian scheme (A, p_A) over $W(k)$ lifting (A_k, p_{A_k}) and such that there is an isomorphism $(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (M, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}})$; here $\tilde{v}_\alpha \in \mathcal{T}(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p))[\frac{1}{p}]$ corresponds to \tilde{t}_α via Fontaine's comparison theory of $A_{B(k)}$.*

5.6.1. Theorem. *We assume that $p = 2$ and that the above three conditions 5.6 (i) to (iii) hold. Warning: we do not assume that the pull backs of \mathcal{A}' via geometric points of $\mathcal{N}_{k(v)}$ have all slopes positive. Then with the notations of 5.1, the morphism $y : \text{Spec}(k) \rightarrow \mathcal{N}'$ factors through $\mathcal{N}_{k(v)}$.*

Proof: Due to 5.6 (i) and (ii), the proof of 5.2 applies entirely to give us that the Zariski closure $\tilde{G}_0 = \tilde{G}_0^1$ of $\tilde{G}_{0\eta_0} = \tilde{G}_{0\eta_0}^1$ in $GL(M_0)$ is a reductive subgroup of $GL(M_0)$. So an isomorphism ρ_{1z} as in 5.5 exists and the constructions of μ_0 , q_{R_0} and z_2 can be performed as in 5.3 and 5.4, provided we take (D, p_D) to be the principally quasi-polarized p -divisible group of the principally polarized abelian scheme (A, p_A) over $W(k)$ mentioned in 5.6 (iii). A $W(k_1)$ -valued point z_3 of \mathcal{N} as in 5.4 always exists, cf. [Va5, proof 6.4.1 ii)]. But if the special fibre of A_1 has positive p -rank, then (as $p = 2$) we can not conclude that z_3 and z_2 are the same $W(k_1)$ -valued points of \mathcal{M} . However, as the filtered F -crystals over k_1 associated to A_2 and A_3 coincide to (M_1, F_2^1, ϕ_1) , from Fontaine comparison theory we get that we can identify canonically $(H_{\text{ét}}^1(A_{2B(k_1)}, \mathbb{Q}_p), p_{A_2}) = (H_{\text{ét}}^1(A_{3B(k_1)}, \mathbb{Q}_p), p_{A_3})$; here we denote also by p_{A_2} and p_{A_3} the perfect bilinear forms on $(H_{\text{ét}}^1(A_{2B(k_1)}, \mathbb{Q}_p), p_{A_2}) =$

$(H_{\acute{e}t}^1(A_{3B(k_1)}, \mathbb{Q}_p), p_{A_3})$ defined by p_{A_2} and p_{A_3} . Under this identification we have natural inclusions $pH_{\acute{e}t}^1(A_{2B(k_1)}, \mathbb{Z}_p) \subset H_{\acute{e}t}^1(A_{3B(k_1)}, \mathbb{Z}_p) \subset \frac{1}{p}H_{\acute{e}t}^1(A_{2B(k_1)}, \mathbb{Z}_p)$ of $\text{Gal}(B(k_1))$ -modules (cf. [Va6, second paragraph of the proof of 2.2.3] applied to the p -divisible groups of A_2 and A_3 ; loc. cit. holds with no restrictions on these two p -divisible groups). So the multiplication by p^2 endomorphism of A_2 factors as a composite homomorphism $A_2 \rightarrow A_3 \rightarrow A_2$ and in fact the principally polarized abelian schemes (A_2, p_{A_2}) and $(A_3, p_{A_3}) := z_3^*(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ are naturally $\mathbb{Z}[\frac{1}{p}]$ -isomorphic. So we get that $\tilde{t}_{1\alpha}$ is the de Rham component of a Hodge cycle $\tilde{w}_{2\alpha}$ on A_2 . This implies that $\tilde{t}_{1\alpha}$ is also the de Rham component of a Hodge cycle $\tilde{w}_{0\alpha}$ on $q_{R_0}^*(\mathcal{A}') \times_{R_0} R_0[\frac{1}{p}]$.

Based on 5.6 (ii) and (iii) and due to the existence of ρ_{1z} , there is an isomorphism between $H_{\acute{e}t}^1(A_{3B(k_1)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k_1) = H_{\acute{e}t}^1(A_{B(k_1)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k_1)$ and $H_{\acute{e}t}^1(A_{2B(k_1)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k_1) = H_{\acute{e}t}^1(A_{1B(k_1)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k_1)$ which takes the p -component $\tilde{v}_{3\alpha}$ of the étale component of $\tilde{w}_{3\alpha} := z_3^*(w_{\alpha}^{\mathcal{A}'})$ into the p -component $\tilde{v}_{2\alpha}$ of the étale component of $\tilde{w}_{2\alpha}$. So as $G_{\mathbb{F}_p}$ is a reductive group (cf. 5.6 (i)) and so connected, from Lang's theorem for affine, connected groups over finite fields we get that any torsor of $G_{\mathbb{Z}_p}$ is trivial. So there is an isomorphism

$$(2) \quad (H_{\acute{e}t}^1(A_{3B(k_1)}, \mathbb{Z}_p), (\tilde{v}_{3\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H_{\acute{e}t}^1(A_{2B(k_1)}, \mathbb{Z}_p), (\tilde{v}_{2\alpha})_{\alpha \in \mathcal{J}}).$$

Let $W(k_1) \hookrightarrow \mathbb{C}$ be as in Case 1 of 5.2. Let $L_2 := H_1(A_{2\mathbb{C}}, \mathbb{Z})$ and $L_3 := H_1(A_{3\mathbb{C}}, \mathbb{Z})$. We identify $W = L_3 \otimes_{\mathbb{Z}} \mathbb{Q} = L_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ in such a way that a $\mathbb{G}_{m\mathbb{Q}}(\mathbb{Q})$ -multiple of ψ and v_{α} are the Betti realizations of the principal polarizations of $A_{3\mathbb{C}}$ and $A_{2\mathbb{C}}$ and respectively of the Hodge cycles $\tilde{w}_{3\alpha}$ and $\tilde{w}_{2\alpha}$, $\forall \alpha \in \mathcal{J}$ (cf. proof of 4.2.1). This identification is unique up to an element of $G(\mathbb{Q})$. We have $G(\mathbb{Q}_p) = G(\mathbb{Q})H$ (cf. [Mi2, 4.9]) and $L_3 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ -conjugate to $L \otimes_{\mathbb{Z}} \mathbb{Z}_p = L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ (cf. proof of 4.2.1). From this last sentence and (2) we get the existence of $g \in G(\mathbb{Q})$ such that $g(L_3 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) = L_2 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

So as we have $L_3[\frac{1}{p}] = L_2[\frac{1}{p}]$ (cf. the existence of the mentioned $\mathbb{Z}[\frac{1}{p}]$ -isomorphism) we get that the $B(k_1)$ -valued points of \mathcal{M} defined naturally by z_3 and z_2 are in the same orbit under the action of the element g^{-1} of $G(\mathbb{A}_f^{(p)})$ on \mathcal{M} (see [Va1, 4.1]). So z_2 factors through \mathcal{N}' as z_3 does. So z_2 factors also through \mathcal{N} , cf. 1.4 1). So q_{R_0} factors through \mathcal{N}' and so as in the end of 5.4 we get that y factors through $\mathcal{N}_{k(v)}$. This ends the proof.

§6. Proof of 1.5 and complements

In this Chapter we first prove 1.6 (see 6.1 and 6.2) and then we apply 1.6 2) to provide new examples of Néron models (see 6.3). We end up with few extra complements on integral models (see 6.4). As in §5, k is an algebraically closed field of countable transcendental degree and of characteristic p and the following notations L , $L_{(p)}$, $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$, $E(G, X)$, v , $k(v)$, $O_{(v)}$, $K_p = GSp(L, \psi)(\mathbb{Z}_p)$, $G_{\mathbb{Z}_{(p)}}$, $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, \mathcal{M} , \mathcal{N}' , \mathcal{N} , $(\mathcal{A}', \mathcal{P}_{\mathcal{A}'})$ are as in 1.3.

We start the proof of 1.6 by first showing that 1.6 2) implies 1.6 1). So we assume that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and that one of the conditions (i) to (iii) of 1.6 1) holds. As in the proof of 3.2.1 we argue that $e(v) = 1$. So from 1.6 2) we get that $\text{Sh}_H(G, X)$ has an integral canonical model over $O_{(v)}$ and (as $e(v) = 1$) from 3.1.4 we get

that this model is \mathcal{N}' itself. So \mathcal{N}' is formally smooth and so we have $\mathcal{N} = \mathcal{N}'$. So 1.6 1) holds.

So to prove 1.6 it suffices to prove 1.6 2). So let (G_1, X_1, H_1) and $O(G_1, X_1, p)$ be as in 1.6 2). So $O(G_1, X_1, p)$ is an étale $\mathbb{Z}_{(p)}$ -algebra. Let $G_{1\mathbb{Z}_{(p)}}$ be as in the beginning of §3. A prime v_1 of $E(G_1, X_1)$ dividing p is p -compact for (G_1, X_1) iff the prime v_1^{ad} of $E(G_1^{\text{ad}}, X_1^{\text{ad}})$ divided by v_1 is p -compact for $(G_1^{\text{ad}}, X_1^{\text{ad}})$. We will prove 1.6 2) in the following two main Steps.

6.1. Step 1. In this Step 1 we assume that (G_1, X_1) is a simple, adjoint Shimura pair. We will apply 3.3 to show that we can choose an injective map $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$ as in 3.3 and a prime v of $E(G, X)$ dividing a prime v_1 of $O(G_1, X_1, p)$ such that $\mathcal{N} = \mathcal{N}'$ is the integral canonical model of $\text{Sh}_H(G, X)$ over $O_{(v)}$. If either (G_1, X_1) is of A_n type or (G_1, X_1) has no compact factors and moreover it is of C_n or $D_n^{\mathbb{H}}$ type, then this follows from 3.3 (iv) and 3.2.1. If $p \geq 5$, this follows from [Va1, 6.4.1] and 3.1.4 regardless of the possible choices we could make in the proof of 3.3.

So we now consider the case when p is arbitrary and (G_1, X_1) has compact factors and it is of B_n , C_n , $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type. If $p = 2$ and (G_1, X_1) is of B_n , $D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type, then we also assume that v_1 is p -compact for (G_1, X_1) . As (G_1, X_1) is a simple, adjoint Shimura pair having compact factors, the \mathbb{Q} -rank of $G^{\text{ad}} = G_1$ is 0 and so based on [BHC] we get that $\text{Sh}(G, X)$ is a pro-étale cover of a smooth, projective $E(G, X)$ -scheme. Let $N \in \mathbb{N} \setminus \{1, 2\}$ be relatively prime to p . Let $K(N)$ be the compact, open subgroup of $GSp(L, \psi)(\widehat{\mathbb{Z}})$ acting trivially on L/NL . Let H_0 be a compact, open subgroup of $G(\mathbb{A}_f^{(p)}) \cap K(N)$ such that \mathcal{N}' is a pro-étale cover of \mathcal{N}'/H_0 , cf. 3.1.1 1). We consider a smooth, projective, toroidal compactification $\mathcal{M}^{\text{stc}}/K(N)$ of $\mathcal{M}/K(N)$ such that the abelian scheme defined by the universal principally abelian scheme over $\mathcal{M}/K(N)$ extends to a semiabelian scheme $\mathcal{S}(N)$ over $\mathcal{M}^{\text{stc}}/K(N)$ (cf. [FC, 6.7 of p. 120]). The fibres of $\mathcal{S}(N)$ over points of the complement of $\mathcal{M}/K(N)$ in $\mathcal{M}^{\text{stc}}/K(N)$ are semiabelian scheme which are not abelian schemes.

We show that we can choose $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$ in 3.3 such that the following two things hold:

- (a) the $k(v)$ -scheme $\mathcal{N}_{k(v)}/H_0$ is an open closed subscheme of $\mathcal{N}'_{k(v)}/H_0$;
- (b) the $k(v)$ -scheme $\mathcal{N}_{k(v)}/H_0$ is proper.

Before proving these two properties we show how they imply that $\mathcal{N} = \mathcal{N}'$ and that \mathcal{N}/H_0 is a smooth, projective $O_{(v)}$ -scheme. The connected components of $\mathcal{N}'_{B(\mathbb{F})}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$, cf. [Va1, 3.3.2]. As \mathcal{N} is $G(\mathbb{A}_f^{(p)})$ -invariant (see 3.1.2) and as its special fibre is non-empty (see 4.4), from the previous sentence we get that each connected component \mathcal{C} of the $W(\mathbb{F})$ -scheme $\mathcal{N}'_{W(\mathbb{F})}/H_0$ has smooth points of characteristic p and so $\mathcal{C}_{B(\mathbb{F})}$ is an absolutely irreducible $B(\mathbb{F})$ -variety. Let \mathcal{C}^{stc} be the normalization in the field of fractions of \mathcal{C} of the pull back to $\text{Spec}(W(\mathbb{F}))$ of $\mathcal{M}^{\text{stc}}/K(N)$. It is a normal, integral, projective $W(\mathbb{F})$ -scheme having \mathcal{C} as an open, Zariski dense subscheme. From (a) and (b) we get that either \mathcal{C} is smooth and equal to \mathcal{C}^{stc} or the special fibre of \mathcal{C}^{stc} is not connected. As the theorem on formal functions implies that this last thing can not hold (see [Har, Cor. 11.3 of p. 279]), we get that \mathcal{C} is a smooth, projective $W(\mathbb{F})$ -scheme. So $\mathcal{N} = \mathcal{N}'$ and \mathcal{N}/H_0 is a smooth, projective $O_{(v)}$ -scheme.

We prove (a) and (b) in two Cases depending on the parity of p . We use the notations of the proof of 3.3; so $F_1, \tilde{G}_1, F_2, E_2, I_p, F_1 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I_p} F_{1i}, F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i \in I_p} F_{2i}, i_0 \in I_p, F_3, T_{\mathbb{Z}(p)}, T'_{\mathbb{Q}} = Z^0(G), T_{\mathbb{Z}(p)}^c$ and $\mathcal{B}_{(p)}$ are as in the proof of 3.3. Warning: for both two Cases below we choose F_2 such that $[F_2 : F_1] \gg 0$ and we also choose $T'_{\mathbb{Q}}$ to be $T_{\mathbb{Q}}$.

6.1.1. Case 1. We assume that either $p \geq 3$. So 6.1 (a) holds, cf. 1.5. We now show that for $[F_2 : F_1] \gg 0$ there is no morphism

$$q^{\text{stc}} : \text{Spec}(k[[x]]) \rightarrow \mathcal{C}^{\text{stc}}$$

whose generic fibre factors through $\mathcal{N}_{\mathbb{F}}/H_0$ and such that

$$B := q^{\text{stc}*}(\mathcal{S}(N) \times_{\mathcal{M}/K(N)} \mathcal{C}^{\text{stc}})$$

is a semiabelian scheme over $k[[x]]$ whose special fibre B_k over k is not an abelian variety.

It suffices to show that the assumption that such a q^{stc} exists leads to a contradiction for $[F_2 : F_1] \gg 0$. As in §5 let $k_1 := \overline{k((x))}$ and let $z_1 \in \mathcal{N}_{W(\mathbb{F})}(W(k_1))$ be such that the special fibre of its composite with the quotient morphism $\mathcal{N}_{W(\mathbb{F})} \rightarrow \mathcal{N}_{W(\mathbb{F})}/H_0$ factors through the point $\text{Spec}(k((x))) \rightarrow \mathcal{N}_{\mathbb{F}}/H_0$ defined naturally by q^{stc} . Let $(M_1, F_1^1, \phi_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}), A_1$ and μ_1 be obtained as in Case 1 of 5.2. .

As $p \geq 3$ from 4.2.1 and 4.2.2 1) we get the existence of an isomorphism

$$\rho_1 : (M_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}(p)} W(k_1), (v_{\alpha})_{\alpha \in \mathcal{J}}).$$

We recall that $L_{(p)}$ and $L_{(p)}^*$ are naturally $E_{2(p)}$ -modules and that $\mathcal{B}_{(p)}$ is an $E_{2(p)}$ -subalgebra of $\text{End}(L_{(p)})$ fixed by $G_{\mathbb{Z}(p)}$, cf. 3.3.1. So as $E_{2(p)}$ is an $F_{2(p)}$ -algebra we get that there is a unique direct sum decomposition

$$L_{(p)}^* \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p = \oplus_{i \in I_p} L^i$$

in $G_{\mathbb{Z}_p}$ -modules such that each simple factor F_{2i} of $F_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ acts trivially on $L^j[\frac{1}{p}]$ if $j \neq i$ and there is no element of $L^i[\frac{1}{p}]$ fixed by F_{2i} . This implies that each L^i is self dual with respect to ψ^* . If $i \neq i_0$, the rank of L^i is bounded in terms just of (G_1, X_1) and so independently of F_2 , cf. properties (vi) and (viii) of the proof of 3.3 and the fact that field F_3 depends only on F_1 and not on F_2 . On the other hand, the rank of L^{i_0} goes to infinity when $[F_2 : F_1]$ goes to infinity (cf. (vi) of the proof of 3.3).

We view $G_{W(k_1)}$ naturally as a closed subgroup of $GL(L_{(p)}^* \otimes_{\mathbb{Z}(p)} W(k_1))$. We recall (see 4.3.1) that we denote by ψ^* the perfect alternating form on $L_{(p)}^*$ and so also on $L_{(p)}^* \otimes_{\mathbb{Z}(p)} W(k_1)$ defined by ψ ; it is normalized by $G_{W(k_1)}$. Let $\mu^0 := \rho_1 \mu_1 \rho_1^{-1} : \mathbb{G}_{mW(k_1)} \rightarrow G_{W(k_1)}$. So $\rho_1 \phi_1 \rho_1^{-1}$ is a σ_{k_1} -linear endomorphism of $L_{(p)}^* \otimes_{\mathbb{Z}(p)} W(k_1)$ of the form $g \sigma_{k_1} \mu^0(\frac{1}{p})$, where $g \in G_{\mathbb{Z}(p)}(W(k_1))$ and σ_{k_1} is identified here with the σ_{k_1} -linear automorphism of $L_{(p)}^* \otimes_{\mathbb{Z}(p)} W(k_1)$ fixing $L_{(p)}^* \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p$. From the second paragraph of Case

1 of 5.2 we get that under any fixed $O_{(v)}$ -embedding $W(k_1) \hookrightarrow \mathbb{C}$, the cocharacters $\mu_{\mathbb{C}}^0$ and μ_h 's of $G_{\mathbb{C}}$ are $G(\mathbb{C})$ -conjugate (here $h \in X$).

We recall from the proof of 3.3 that the centralizer of $G_{\mathbb{Z}_{(p)}}$ in $GL(L_{(p)})$ is a reductive group scheme $C_{\mathbb{Z}_p}$. So the centralizer of $G_{\mathbb{Z}_p}$ in $GL(L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ is also a reductive group scheme and so has a maximal torus $T(C_{\mathbb{Z}_p})$ which is the group scheme of invertible elements of a commutative, semisimple \mathbb{Z}_p -algebra $F(C_{\mathbb{Z}_p})$. As $\mathcal{B}_{(p)}$ is an $E_{2(p)}$ -algebra, we get that $F(C_{\mathbb{Z}_p})$ is in fact an $E_{2(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ -semisimple algebra. Let J_0 be the set of distinct characters of the action of $T(C_{\mathbb{Z}_p})_{W(k_1)}$ on $L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1)$. The Frobenius automorphism σ_{k_1} acts naturally on J_0 . Let M^j be the direct summand of $L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1)$ on which $T(C_{\mathbb{Z}_p})_{W(k_1)}$ acts via the character $j \in J_0$. We get a direct sum decomposition

$$L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1) = \bigoplus_{j \in J_0} M^j$$

into irreducible $G_{W(k_1)}$ -modules permuted by σ_{k_1} .

We fix an orbit o of the action of σ_{k_1} on J_0 . As $T_{\mathbb{Q}} = T'_{\mathbb{Q}}$, the torus $T_{W(k_1)}^c$ is a subtorus of $G_{W(k_1)}$. The action of $T_{W(k_1)}^c$ on $L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1)$ is via two non-trivial and distinct irreducible cocharacters. So there is a disjoint decomposition

$$J_0 = J_1 \cup J_2$$

such that for j_1 and $j_2 \in J_0$, the $T_{W(k_1)}^c$ -modules M^{j_1} and M^{j_2} are isomorphic iff $(j_1, j_2) \in J_1^2 \cup J_2^2$. As $T_{W(k_1)}^c$ fixes ψ , for $(j_1, j_2) \in J_1^2 \cup J_2^2$ we have $\psi^*(M^{j_1}, M^{j_2}) = \{0\}$.

There is a simple factor of the adjoint group of the image of $G_{W(k_1)}$ in $GL(L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1))$ in which μ^0 has a trivial image. This is implied by the fact that $i_0 \in I_p$ is a compact element in the sense of the proof of 3.3 and by the $G(\mathbb{C})$ -conjugacy property of the last sentence of the third paragraph above. So there is $j_0 \in o$ such that the image of μ^0 in $GL(M^{j_0})$ is contained in $Z(GL(M^{j_0}))$. As $E_2 \otimes_{F_2} F_{2i_0}$ is a quadratic field extension of F_{2i_0} and as $F(C_{\mathbb{Z}_p})[\frac{1}{p}]$ is an $E_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ -algebra, the orbit o contains at least two characters $j_1 \in J_1$ and $j_2 \in J_2$ such that M^{j_1} and M^{j_2} are not perpendicular with respect to ψ^* . So there is $i \in \{1, 2\}$ and $j_i \in o \cap J_i$, such that the image of μ^0 in $GL(M^{j_i})$ is trivial. So the multiplicity of the slope 1 for $(L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1), \tilde{g}g\sigma_{k_1}\mu^0(\frac{1}{p}))$ is 0 for any $\tilde{g} \in G_{W(k_1)}(W(k_1))$. So as L^{i_0} is self dual with respect to ψ^* , the multiplicity of the slope 0 for $(L^{i_0} \otimes_{\mathbb{Z}_p} W(k_1), \tilde{g}g\sigma_{k_1}\mu^0(\frac{1}{p}))$ is also 0 for any $\tilde{g} \in G_{W(k_1)}(W(k_1))$.

So the p -rank of A_{1k_1} (and so also of the generic fibre $B_{k((x))}$ of B) is at most

$$n_0 := \frac{1}{2} \sum_{i \in I_p \setminus \{i_0\}} \dim_{\mathbb{Z}_p}(L^i)$$

and so bounded above just in terms of (G_1, X_1) and so independently of $[F_2 : F_1]$.

Let $T(B_k)$ be the toric part (i.e. the maximal torus) of the identity component of B_k . As B_k is not an abelian scheme, we have $1 \leq \dim_k(T(B_k))$. From [FC, Cor. 5.11 of p. 70] we get that $\forall m \in \mathbb{N}$ the finite group scheme $T(B_k)[p^m]_{k((x))}$ is naturally identified with a closed subgroup of $B_{k((x))}[p^n]$ and that this identification is compatible with the

$F_{2(p)}$ -actions. So the p -rank of A_{1k_1} (and so also of $B_{k((x))}$) is at least $\dim_k(T(B_k))$. So we have $\dim_k(T(B_k)) \leq n_0$. But $F_{2(p)}$ acts on B and so also on the \mathbb{Z} -module $X_*(T(B_k))$ of cocharacters of $T(B_k)$. So as $1 \leq \dim_{\mathbb{Z}}(X_*(T(B_k))) \leq n_0$, by taking $[F_2 : F_1] > n_0$ we get that this action is trivial. So $F_{2(p)}$ acts trivially on $T(B_k)$. As $T_0[p^m]_{k((x))}$ is naturally identified with a closed subgroup of $B_{k((x))}[p^m]$ and as this identification is compatible with the $F_{2(p)}$ -actions, we get that $F_{2(p)}$ acts trivially on a direct summand of M_1 of positive rank. So via ρ_1 , $F_{2(p)}$ acts trivially on a direct summand of $L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k_1)$ of positive rank. But this contradicts 3.3.1. So a morphism q^{stc} as in the beginning of 6.1.1 does not exist. This implies that 6.1 (b) holds for $p \geq 3$.

6.1.2. Case 2. Let $p = 2$. Let $(T_{\mathbb{Q}}^0, \{h\})$, H^0 and \mathcal{T}^0 be as in the proof of 4.4 (i). We fix a point $z \in \text{Im}(\mathcal{T}^0(W(\mathbb{F})) \rightarrow \mathcal{N}(W(\mathbb{F})))$. The statements of 4.2.2 1) and 2) hold for $k = \mathbb{F}$ and for $z \in \mathcal{T}^0(W(\mathbb{F}))$, cf. the toric part of the hypotheses of 4.2.2 applied in the context of the morphism $\mathcal{T}^0 \rightarrow \mathcal{M}$ and so of the injective map $(T_{\mathbb{Q}}^0, \{h\}) \hookrightarrow (GSp(W, \psi), S)$ of Shimura pairs. Let $\mathcal{N}_{k(v)}^0$ be the connected component of $\mathcal{N}_{k(v)}$ to which z specializes, cf. 1.4 1). Let H_{00} be the maximal closed subgroup of H_0 normalizing $\mathcal{N}_{k(v)}^0$.

If $q_R : \text{Spec}(R) \rightarrow \mathcal{N}$ is as in 4.3.3 and if $\mathcal{C}_{\text{univ}}$ is as in 4.6, then the pull back of $(M_R, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}})$ via any $W(k_1)$ -valued point of $\text{Spec}(R)$, is isomorphic to $(M \otimes_{W(k)} W(k_1), (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}})$. Moreover the pull back of $w_{\alpha}^{A'}$ via $q_{R[\frac{1}{p}]}$ is a Hodge cycle on $A_{R[\frac{1}{p}]}$ having $\tilde{t}_{\alpha} \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$ as its de Rham component, cf. 4.3.3. So as 4.2.2 1) and 2) hold for z and due to 4.2.1, we get that the logical analogues of 4.2.2 1) and 2) also hold for any $W(k_1)$ -valued point of \mathcal{N} factoring through the formally smooth morphism $q_R : \text{Spec}(R) \rightarrow \mathcal{N}$.

We take \mathcal{C} to be the connected component of $\mathcal{N}'_{W(\mathbb{F})}/H_0$ whose special fibre $\mathcal{C}_{\mathbb{F}}$ contains $\mathcal{N}_{\mathbb{F}}^0/H_{00}$. Let $q^{\text{stc}} : \text{Spec}(k[[x]]) \rightarrow \mathcal{C}^{\text{stc}}$ be a dominant morphism; so its generic fibre factors through a generic point of $\mathcal{N}_{\mathbb{F}}^0/H_{00}$. We consider two Subcases and we use the notations of Case 1.

6.1.2.1. Subcase 1. We consider the Subcase when v_1 is p -compact for (G_1, X_1) . The prime v_1 is p -compact iff each $i \in I_p$ is a compact element. So as in Case 1 we argue that $(L^i, g\sigma_{k_1}\mu^0(\frac{1}{p}))$ has no slope 1, $\forall i \in I_p$. So also (M_1, ϕ_1) has no slope 1. So the p -rank of A_{1k_1} is 0. So as q^{stc} is dominant, the pull backs of \mathcal{A}' via geometric points of $\mathcal{N}_{k(v)}^0$ have p -ranks equal to 0. So from the proof of 1.5 applied just to the connected component $\mathcal{N}_{k(v)}^0$ of $\mathcal{N}_{k(v)}$, we get that $\mathcal{N}_{k(v)}^0$ is an open closed subscheme of $\mathcal{N}'_{k(v)}$. Moreover, either from loc. cit. or from [Oo, 1.3] we get that A_{1k_1} can not specialize to a semiabelian variety which is not an abelian variety. So $\mathcal{N}_{\mathbb{F}}^0/H_{00}$ is a smooth, open closed subscheme of $\mathcal{C}_{\mathbb{F}}^{\text{stc}}$. So as in Case 1 we argue that $\mathcal{C} = \mathcal{C}^{\text{stc}}$ is smooth and projective. So as the connected components of \mathcal{N} or \mathcal{N}' are permuted transitively by $G(\mathbb{A}_f^{(p)})$ (cf. [Va1, 3.2.2]), we get that $\mathcal{N} = \mathcal{N}'$ and that $\mathcal{N}_{k(v)}/H_0$ is a proper scheme. So also 6.1 (a) and (b) hold in this case.

6.1.2.2. Subcase 2. We consider the Subcase when (G_1, X_1) is of C_n type and has compact factors. We will first use 5.6.1 to show that $\mathcal{N}_{k(v)}^0$ is an open closed subscheme of $\mathcal{N}'_{k(v)}$.

Let $q : \text{Spec}(k[[x]]) \rightarrow \mathcal{N}'_{k(v)}$ be a morphism whose generic fibre factors through the

generic point of $\mathcal{N}_{k(v)}^0$. Let $R_0 = W(k)[[x]]$, Φ_{R_0} , $(M_0, \Phi_0, \nabla_0, p_{M_0})$ and $(\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}}$ be as in 5.1. Let $G_{\mathbb{Z}_p}^1 := G_{\mathbb{Z}_p}$. Let $\tilde{G}_0 = \tilde{G}_0^1$ be as in 5.1.2; it is a reductive group scheme over R_0 (cf. 5.2). Let $(M_1, F_1^1, \phi_1, (\tilde{t}_{1\alpha})_{\alpha \in \mathcal{J}})$, A_1 , μ_1 , $\mu_0 : \mathbb{G}_{mR_0} \rightarrow \tilde{G}_0$ and $(M, F^1, \phi, \tilde{G}, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}}, p_M)$ be as in Case 1 of 5.2 and 5.3. Let y be as in 5.1 and let (A_k, p_{A_k}) be as in 5.4.

In order to apply 5.6.1, we will check that conditions 5.6 (i) to (iii) hold. The group scheme $G_{\mathbb{Z}_{(p)}}$ is reductive (cf. 3.3 (ii)) and we have $G_{\mathbb{Z}_p}^1 = G_{\mathbb{Z}_p}$; so 5.6 (i) holds. The point $z_1 : \text{Spec}(W(k_1)) \rightarrow \mathcal{N}$ is such that its generic fibre factors through the generic point of $\mathcal{N}_{k(v)}^0$ and so implicitly through the special fibre $q_{R/pR}$ of q_R . So 4.2.2 2) applies in the context of z_1 (see beginning of 6.1.2) and so 5.6 (ii) holds.

We now check that 5.6 (iii) holds. Let ρ_1 be as in Case 1. As 5.6 (i) and (ii) hold, from the first paragraph of the proof of 5.6.1 we get that \tilde{G}_0 is a reductive subgroup of $GL(M_0)$. So let ρ_{1z} be as in 5.5. So $\rho_1 \rho_{1z}^{-1}$ is an isomorphism $(M \otimes_{W(k)} W(k_1), (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k_1), (v_{\alpha})_{\alpha \in \mathcal{J}})$. As $k = \bar{k}$, we get the existence of an isomorphism $\rho : (M, (\tilde{t}_{\alpha})_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k), (v_{\alpha})_{\alpha \in \mathcal{J}})$. Let $M_i := \rho^{-1}(L^i \otimes_{\mathbb{Z}_p} W(k))$. Let \tilde{G} be the reductive subgroup of $GL(M)$ which is the closed subscheme of \tilde{G}_0 defined by the zero locus $x = 0$. We get a direct sum decomposition of \tilde{G} -modules

$$M = \oplus_{i \in I_p} M_i$$

We choose the principally polarized abelian scheme (A, p_A) over $W(k)$ lifting (A_k, p_{A_k}) such that the principally quasi-polarized p -divisible group (D, p_D) of (A, p_A) is a direct sum $\oplus_{i \in I_p} (D_i, p_{D_i})$, where the filtered F -crystal of D_i is $(M_i, F^1 \cap M_i, \phi)$. Implicitly $H_{\acute{e}t}^1(A_{B(k)}, \mathbb{Z}_p)$ is a direct sum $\oplus_{i \in I_p} H^{1i}$, where H^{1i} is the dual of the Tate-module of D_i . As we have $G_{\mathbb{Q}_p}^{\text{ad}} = G_{1\mathbb{Q}_p} = \prod_{i \in I_p} \text{Res}_{F_{1i}/\mathbb{Q}} \tilde{G}_{1F_{1i}}$, we can define the closed, flat subgroup $G_{\mathbb{Z}_p}^{\text{id}}^{\text{der}}$ of $GL(H_{\acute{e}t}^1(A_{B(k)}, \mathbb{Z}_p))$ whose generic fibre corresponds to the semisimple subgroup of $G_{\mathbb{Q}_p}^{\text{der}}$ having $\text{Res}_{F_{1i}/\mathbb{Q}} \tilde{G}_{1F_{1i}}$ as its adjoint, via the Fontaine comparison theory of $A_{B(k)}$. Also, we can define the closed, flat subgroup $Z_{\mathbb{Z}_p}^0$ of $GL(H_{\acute{e}t}^1(A_{B(k)}, \mathbb{Z}_p))$ whose generic fibre corresponds to $Z^0(G_{\mathbb{Q}_p})$ via the same theory.

We view naturally $\mathcal{B}_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ as a \mathbb{Z}_p -algebra of \mathbb{Z}_p -endomorphisms of \mathcal{A}' . So $\mathcal{B}_{(p)}^{\text{opp}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ acts naturally on M . We consider the product decomposition $\mathcal{B}_{(p)}^{\text{opp}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = \prod_{i \in I_p} \mathcal{B}_i$ such that $\mathcal{B}_i[\frac{1}{p}]$ is an F_i -algebra. So \mathcal{B}_i acts faithfully on M_i and fixes M/M_i . We show that $G_{\mathbb{Z}_p}^{\text{id}}^{\text{der}}$ is a semisimple group scheme over \mathbb{Z}_p , $\forall i \in I_p$.

If $i \in I_p$ is a compact element, then as in Case 1 we argue that $(M_i, \tilde{g}\phi)$ has no slope 1, $\forall \tilde{g} \in \tilde{G}(W(k))$. So (D_i, p_{D_i}) is uniquely determined, cf. [Va6, 2.2.3]. Let $(t_{\alpha})_{\alpha \in \mathcal{J}_i}$ be a family of tensors of $\mathcal{T}(M_i)$ fixed by ϕ and such that the image of \tilde{G} in $GL(M_i)$ is the Zariski closure of the subgroup of $GL(M_i[\frac{1}{p}])$ fixing t_{α} , $\forall \alpha \in \mathcal{J}_i$. Its existence is implied by [Va6, 2.5.3]. From [Va6, 1.3 a)] applied to $(D_i, (t_{\alpha})_{\alpha \in \mathcal{J}_i})$ we get that $G_{W(k)}^{\text{id}}^{\text{der}}$ is isomorphic to the image of \tilde{G}^{der} in $GL(M_i)$ and so $G_{\mathbb{Z}_p}^{\text{id}}^{\text{der}}$ is a semisimple group scheme.

We now consider an $i \in I_p$ which is not a compact element. Let \tilde{G}_0^i be the subgroup of $GL(M_i)$ generated by the image of \tilde{G}^{der} in $GL(M_i)$ and by $Z(GL(M_i))$. It is a reductive group scheme, cf. 2.2.5 2). Let p_{M_i} be the perfect alternating form on M_i defined by

p_M . Let μ be the cocharacter of \tilde{G} which is the reduction modulo x of the cocharacter $\mu_0 : \mathbb{G}_{mR_0} \rightarrow \tilde{G}_0$ of 5.3. Let $\mu_i : \mathbb{G}_{mW(k)} \rightarrow GL(M_i)$ be the cocharacter defined by μ . As $i \in I_p$ is not a compact element, μ_i factors through \tilde{G}_0^i . The cocharacter $\mu_0 : \mathbb{G}_{mR_0} \rightarrow \tilde{G}_0$ is uniquely determined up to inner conjugation under an element $g_0 \in \tilde{G}_0(R_0)$ which normalizes the kernel F_0^1/pF_0^1 of $\Phi_0 \bmod p$. Let S^i be the semisimple \mathbb{Z}_p -algebra of elements of $\text{End}(M_i)$ fixed by \tilde{G}_0^i and by ϕ . We identify its opposite $S^{i\text{opp}}$ with a \mathbb{Z}_p -algebra of endomorphisms of the special fibre D_{ik} of D_i . The subgroup of $GSp(M_i, p_{M_i})$ fixing $S^i \otimes_{\mathbb{Z}_p} W(k)$ is \tilde{G}_0^i . As all simple factors of the adjoint group of $\tilde{G}_0^{\text{idder}}$ are of C_n Lie type, it is well known that the formal moduli space of the pair $((D_i, p_{D_i})_k, S^{i\text{opp}})$ is formally smooth and representable by a scheme of formal power series over $W(k)$.

So the pair $((D_i, p_{D_i})_k, S^{i\text{opp}})$ has a lift $((D'_i, p_{D'_i}), S^{i\text{opp}})$ to $\text{Spec}(W(k))$. Let $F^{1'}$ be the direct summand of M_i which is the Hodge filtration of D'_i . Let $\mu'_i : \mathbb{G}_{mW(k)} \rightarrow GL(M_i)$ be the inverse of the canonical split cocharacter of $(M_i, F^{1'}, \phi)$ defined in [Wi, p. 512]. As the group \tilde{G}_0^i is the subgroup of $GL(M_i, p_{M_i})$ centralizing $S^i \otimes_{\mathbb{Z}_p} W(k)$ and as μ'_i fixes each element of S^i and normalizes p_{M_i} (cf. the functorial aspects of [Wi, p. 513]), μ'_i factors through \tilde{G}_0^i . So μ'_i is conjugate with μ_i under an element $g_0^i \in \tilde{G}_0^i(W(k))$ normalizing $F^1/pF^1 \cap M_i/pM_i$, cf. [Va5, 3.1.2]. It is easy to see that we can choose the g_0^i 's for $i \in I_p$ not a compact element, such that there is $g_0 \in \tilde{G}_0(R_0)$ normalizing F_0^1/pF_0^1 and with the property that its image in $\tilde{G}_i^0(W(k))$ is g_0^i , for any $i \in I_p$ which is not a compact element. Not to introduce extra notations we will assume that $\mu_i = \mu'_i$, for any $i \in I_p$ which is not a compact element.

So we can also assume that $(D_i, p_{D_i}) = (D'_i, p_{D'_i})$. We denote also by S^i the \mathbb{Z}_p -algebra of endomorphisms of H^{1i} corresponding to the endomorphisms $S^{i\text{opp}}$ of $D_i = D'_i$. Let $p_{H^{1i}}$ be the perfect alternating form on H^{1i} which is the étale realization of $p_{D_i} = p_{D'_i}$ (equivalently of p_{M_i}). As (G_1, X_1) is of C_n type, it is well known that the subgroup of $GL(H^{1i})$ fixing S^i and normalizing $p_{H^{1i}}$ is reductive. Its derived group is $G_{\mathbb{Z}_p}^{\text{idder}}$.

We conclude that $G_{\mathbb{Z}_p}^{\text{idder}}$ is a semisimple group scheme over \mathbb{Z}_p , $\forall i \in I_p$. The group scheme $Z_{\mathbb{Z}_p}^0$ is the Zariski closure of a subtorus of the generic fibre of the torus of $GL(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p))$ which is the center of the centralizer of $\prod_{i \in I_p} \mathcal{B}_i$ in $GL(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p))$. So $Z_{\mathbb{Z}_p}^0$ is a torus. From 2.2.5 2) we get that the Zariski closure $G_{\mathbb{Z}_p}^{\text{ét}}$ in $GL(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p))$ of the subgroup of $GL(H_{\text{ét}}^1(A_{B(k)}/\mathbb{Q}_p))$ fixing \tilde{v}_α , $\forall \alpha \in \mathcal{J}$, is reductive.

The center of the centralizer of $\prod_{i \in I_p} \mathcal{B}_i$ in $GSp(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p), \oplus_{i \in I_p} p_{H^{1i}})$ is $Z_{\mathbb{Z}_p}^0$, cf. 3.3.2. So as in [Va5, 7.8.3.1] (working under the assumptions [Va5, 7.8.3 1) and 2a)]) we get that there is an isomorphism $(M, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}})$. So 5.6 (iii) holds.

So all hypotheses of 5.6.1 hold and so from 5.6.1 we get that the morphism $q : \text{Spec}(k[[x]]) \rightarrow \mathcal{N}'_{k(v)}$ factors through $\mathcal{N}_{k(v)}$. This implies that $\mathcal{N}_{k(v)}^0$ is an open closed subscheme of $\mathcal{N}'_{k(v)}$. As in Case 1 we check that for $[F_2 : F_1] \gg 0$, there is no morphism $q^{\text{stc}} : \text{Spec}(k[[x]]) \rightarrow \mathcal{C}^{\text{stc}}$ whose generic fibre factors through $\mathcal{N}_{\mathbb{F}}^0/H_0$ and such that the natural pull back of $\mathcal{S}(N)$ to $\text{Spec}(k[[x]])$ is a semiabelian scheme which is not an abelian scheme. The rest of the arguments needed to prove 6.1 (a) and (b) are as in Subcase 1. So 6.1 (a) and (b) hold for $p = 2$ as well.

6.1.3. More on the A_n type. We assume (G_1, X_1) is a simple, adjoint Shimura pair of A_n type and with compact factors. Let F_2 and (G_2, X_2) be defined as in the proof of 3.3. We apply 3.3 to (G_2, X_2) instead of (G_1, X_1) and we denote also by $f : (G, X) \hookrightarrow (GSp(W, \psi), S)$ the resulting embedding. So to end this Step 1, we point out that entirely as above (for the B_n , C_n , $D_n^{\mathbb{H}}$ and $D_n^{\mathbb{R}}$ types) we argue that for $[F_2 : F_1] \gg 0$ the $O_{(v)}$ -scheme \mathcal{N}/H_0 is smooth and projective.

6.2. Step 2. In this Step 2 we prove 1.6 2) in general. So we work with a general Shimura quadruple (G_1, X_1, H_1, v_1) satisfying the hypotheses of 1.6 2). If G_1 is a torus, then the fact that $\text{Sh}_{H_1}(G_1, X_1)$ has an integral canonical model over $O_{(v_1)}$ is well known (for instance, see [Va1, 3.2.8]). So for the rest of the proof (i.e. until 6.3) we will assume that the group G_1^{ad} is non-trivial. Let $G_{1\mathbb{Z}_{(p)}}$ be the reductive group scheme over $\mathbb{Z}_{(p)}$ extending G_1 and such that H_1 is its group scheme of \mathbb{Z}_p -valued points, cf. beginning of §3.

Let $H_1^{\text{ad}} := G_{1\mathbb{Z}_{(p)}}^{\text{ad}}(\mathbb{Z}_p)$. Let v_1^{ad} be the prime of $E(G_1^{\text{ad}}, X_1^{\text{ad}})$ divided by v_1 . We refer to the Shimura quadruple $(G_1^{\text{ad}}, X_1^{\text{ad}}, H_1^{\text{ad}}, v_1^{\text{ad}})$ as the adjoint Shimura quadruple of (G_1, X_1, H_1, v_1) . We consider its product decomposition

$$(G_1^{\text{ad}}, X_1^{\text{ad}}, H_1^{\text{ad}}, v_1^{\text{ad}}) = \prod_{i \in I_0} (G_1^i, X_1^i, H_1^i, v_1^i)$$

into simple factors. So each G_1^i is a simple, adjoint \mathbb{Q} -group extending to an adjoint group scheme $G_{1\mathbb{Z}_{(p)}}^i$ over $\mathbb{Z}_{(p)}$ such that we have $G_{1\mathbb{Z}_{(p)}}^{\text{ad}} = \prod_{i \in I_0} G_{1\mathbb{Z}_{(p)}}^i$.

For each $i \in I_0$ we apply 3.3 and 6.1. So we consider an injective map $f^i : (G^i, X^i) \hookrightarrow (GSp(W^i, \psi^i), S^i)$ such that there is a \mathbb{Z} -lattice L^i of W^i with the property that we have a perfect alternating form $\psi^i : L^i \otimes_{\mathbb{Z}} L^i \rightarrow \mathbb{Z}$, the analogues of 3.3 (i) to (iv) hold for f^i but with (G_1, X_1) being replaced by (G_1^i, X_1^i) and moreover (cf. 6.1) the integral canonical model \mathcal{N}^i of $\text{Sh}_{H^i}(G^i, X^i)$ over $O(G_1^i, X_1^i, p)$ exists and is obtained in the same \mathcal{N}' was obtained in 1.3. Here H^i is the hyperspecial subgroup of $G_{\mathbb{Q}_p}^i$ which is the group of \mathbb{Z}_p -valued points of the Zariski closure of G^i in $GL(L^i \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

Let $(G_3, X_3) \rightarrow (G_1, X_1)$ be a map of Shimura pairs such that the following five properties hold (cf. [MS, proof of 3.4] and [Va1, 3.2.7–10]):

- (i) *it is a cover in the sense of [Va1, 2.4], i.e. G_3 surjects onto G_1 and $\text{Ker}(G_3 \rightarrow G_1)$ is a torus such that $H^1(K, \text{Ker}(G_3 \rightarrow G_1)) = \{0\}$, for any field K of characteristic 0;*
- (ii) *we have $E(G_3, X_3) = E(G_1, X_1)$;*
- (iii) *the group G_3^{der} is simply connected if $p > 2$ and is $\prod_{i \in I_0} G^{i\text{der}}$ if $p = 2$;*
- (iv) *the group $G_{3\mathbb{Q}_p}$ is unramified;*
- (v) *we have a product decomposition $(G_3, X_3) = \prod_{i \in I_0} (G_3^i, X_3^i)$ such that the adjoint of (G_3^i, X_3^i) is (G_1^i, X_1^i) .*

Let $G_{3\mathbb{Z}_{(p)}}$ be the reductive group scheme over $\mathbb{Z}_{(p)}$ extending G_3 and having $G_{1\mathbb{Z}_{(p)}}^{\text{ad}}$ as its adjoint. Let $H_3 := G_{3\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$.

Let $i \in I_0$. If $p = 2$ or if (G_1^i, X_1^i) is not of $D_n^{\mathbb{H}}$ type, then let $(G^{i0}, X^{i0}) := (G^i, X^i)$ and $\mathcal{N}^{i0} := \mathcal{N}^i$. If $p > 2$ and (G_1^i, X_1^i) is of $D_n^{\mathbb{H}}$ type, then we perform the following construction. As above we consider a cover $(G^{i0}, X^{i0}) \rightarrow (G^i, X^i)$ with $G_{\mathbb{Q}_p}^{i0}$ unramified, with

$G^{i0\text{der}}$ simply connected and with $E(G^{i0}, X^{i0}) = E(G^i, X^i)$. Let H^{i0} be the hyperspecial subgroup of $G_{\mathbb{Q}_p}^{i0}(\mathbb{Q}_p)$ which is the analogue of H_3 but obtained in the context of the cover $(G^{i0}, X^{i0}) \rightarrow (G^i, X^i)$. As $p > 2$, the logical analogue of 4.2.2 2) always holds for any $z \in \mathcal{N}^i(W(k))$. So the triple (f^i, L^i, v^i) is a standard Hodge situation in the sense of [Va5, 1.4], for any prime v^i of $E(G^i, X^i)$ dividing p . So the integral canonical model \mathcal{N}^{i0} of $\text{Sh}_{H^{i0}}(G^{i0}, X^{i0})$ over $O(G^{i0}, X^{i0}, p) = O(G^i, X^i, p)$ exists and is a pro-étale cover of \mathcal{N}^i , cf. [Va5, 7.9.3].

So regardless of who p and (G_1^i, X_1^i) 's are we get:

(vi) *we have $O(G^{i0}, X^{i0}, p) = O(G^i, X^i, p)$ and \mathcal{N}^{i0} is a pro-étale cover of a smooth, (quasi-) projective $O(G^{i0}, X^{i0}, p)$ -scheme iff \mathcal{N}^i is a pro-étale cover of a smooth, (quasi-) projective $O(G^i, X^i, p)$ -scheme.*

Let $(G_4, X_4, H_4) := \prod_{i \in I_0} (G^{i0}, X^{i0}, H^{i0})$. The product of the $O(G_4, X_4, p)$ -schemes $\mathcal{N}_{O(G_4, X_4, p)}^{i0}$'s is the integral canonical model \mathcal{N}_4 of $\text{Sh}_{H_4}(G_4, X_4)$ over $O(G_4, X_4, p)$. We have $G_4^{\text{der}} = G_3^{\text{der}}$. So as in [Va1, 6.2.3] we argue that:

(vii) *$\text{Sh}_{H_3}(G_3, X_3)$ has an integral canonical model \mathcal{N}_3 over $O(G_3, X_3, p)$ and that for any pair of primes v_4 and v_3 of $O(G_4, X_4, p)$ and respectively of $O(G_3, X_3, p)$ dividing the same prime of $O(G_1^{\text{ad}}, X_1^{\text{ad}}, p) = O(G_4^{\text{ad}}, X_4^{\text{ad}}, p) = O(G_3^{\text{ad}}, X_3^{\text{ad}}, p)$, the connected components of $\mathcal{N}_{4O^{\text{sh}}(v_4)}$ and $\mathcal{N}_{3O^{\text{sh}}(v_3)}$ are isomorphic.*

From (vii) we get

(viii) *\mathcal{N}_3 is a pro-étale cover of a smooth, (quasi-) projective $O(G_3, X_3, p)$ -scheme iff \mathcal{N}_4 is a pro-étale cover of a smooth, (quasi-) projective $O(G_4, X_4, p)$ -scheme.*

From 3.1.1 4) and 3.1.5 we also we get that:

(ix) *\mathcal{N}^i is a pro-étale cover of a smooth, quasi-projective $O(G^i, X^i, p)$ -scheme.*

We now check the following two things:

(a) *the integral canonical model $\mathcal{N}_1^{\text{ad}}$ of $\text{Sh}_{H_1^{\text{ad}}}(G_1^{\text{ad}}, X_1^{\text{ad}})$ over $O(G_1^{\text{ad}}, X_1^{\text{ad}}, p)$ exists and is a pro-étale cover of a smooth, quasi-projective $O(G_1^{\text{ad}}, X_1^{\text{ad}}, p)$ -scheme;*

(b) *if (G_1^i, X_1^i) has compact factors for all $i \in I_0$, then $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, projective $O(G_1^{\text{ad}}, X_1^{\text{ad}}, p)$ -scheme.*

The same argument showing that \mathcal{N}_4 exists, shows that it suffices to prove (a) and (b) under the extra assumption that I_0 has only one element i . So (G_1, X_1) is a simple, adjoint Shimura pair and so we can appeal to the notations of 6.1. However, for the sake of uniformity of notations we will continue to use the notation $\mathcal{N}_1^{\text{ad}}$ instead of \mathcal{N}_1 . As I_0 has only one element i , we have $\mathcal{N}_4 = \mathcal{N}^{i0}$. To check (a) and (b) we consider three Cases.

6.2.1. Case 1: (G_1, X_1) is of A_n type. As 3.3 (iv) holds for F^i and L^i , the triple (f^i, L^i, v^i) is a standard PEL situation for any prime v^i of $E(G^i, X^i)$ dividing p . So from [Va4, 5.3.3] we get that $\mathcal{N}_1^{\text{ad}}$ exists and that \mathcal{N}^i is a pro-étale cover of a non-empty, open closed subscheme of $\mathcal{N}_1^{\text{ad}}$. From this and 6.2 (ix) we get that a non-empty, open closed subscheme of $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme.

So as the connected components of $\mathcal{N}_1^{\text{ad}}$ are permuted transitively by $G_1(\mathbb{A}_f^{(p)})$ (cf. [Va1, 3.3.2]), we get that $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme. So 6.2 (a) holds. If $(G_1, X_1) = (G_1^i, X_1^i)$ has compact factors, then we can choose f^i such that \mathcal{N}^i is a pro-étale cover of a projective $O(G^i, X^i, p)$ -scheme (cf. 6.1.3). So similarly we argue that $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, projective $O(G_1, X_1, p)$ -scheme; so 6.2 (b) also holds.

6.2.2. Case 2: $p \geq 3$ and (G_1, X_1) is of $B_n, C_n, D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type. In this Case the order of the center of $G^{\text{sc}} = G_1^{\text{sc}}$ is a power of 2 and so prime to p . So from [Va1, 6.2.2 a)] applied in the context of the cover $(G_3, X_3, H_3) \rightarrow (G_1, X_1, H_1)$ we get that $\mathcal{N}_1^{\text{ad}}$ exists and that \mathcal{N}_3 is a pro-étale cover of $\mathcal{N}_1^{\text{ad}}$. From 6.2 (vii), (viii) and (ix) we get that \mathcal{N}_3 is a pro-étale cover of a smooth, quasi-projective $O(G_3, X_3, p)$ -scheme. From the last two sentences we get that $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme and so that 6.2 (a) holds. If $(G_1, X_1) = (G_1^i, X_1^i)$ has compact factors, then from 6.1 we get that we can choose f^i such that \mathcal{N}^i is a pro-étale cover of a projective $O(G^i, X^i, p)$ -scheme. So based on 6.2 (vi) and (viii) and the equality $\mathcal{N}_4 = \mathcal{N}^{i0}$ we get that 6.2 (b) holds.

6.2.3. Case 3: $p = 2$ and (G_1, X_1) is of $B_n, C_n, D_n^{\mathbb{H}}$ or $D_n^{\mathbb{R}}$ type. Let $\tilde{G}'_{\mathbb{Z}_{(p)}}$ be as in 3.3.2 and 3.3.3. Let $\tilde{G}' := \tilde{G}'_{\mathbb{Q}}$. Let \tilde{X}' be such that the monomorphism $G \hookrightarrow \tilde{G}'$ extends to an injective map $(G, X) \hookrightarrow (\tilde{G}', \tilde{X}')$ of Shimura pairs. Let $(G'_2, X'_2) := (\tilde{G}'^{\text{ad}}, \tilde{X}'^{\text{ad}})$. Let $\tilde{H}' := \tilde{G}'_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and $H'_2 := \tilde{H}'^{\text{ad}} = G'_{2\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Each simple factor of $G'_{2\mathbb{C}}^{\text{ad}}$ is of A_s Lie type, where $s \in \{2n, 2^n, 2^{n-1}\}$ (cf. 3.3.2 and 3.3.3). So based on 6.2.1 we get that the integral canonical model $\tilde{\mathcal{N}}'$ (resp. \mathcal{N}'_2) of $\text{Sh}_{\tilde{H}'}(\tilde{G}', \tilde{X}')$ (resp. of $\text{Sh}_{H'_2}(G'_2, X'_2)$) over the localization of $E(\tilde{G}', \tilde{X}')_{(p)}$ (resp. of $E(G'_2, X'_2)_{(p)}$) with respect to the finite primes divided by the finite primes of $O(G_1, X_1, p)$, exists.

We denote by \tilde{f}' and f'_2 the injective maps $\tilde{f} : (G, X) \hookrightarrow (\tilde{G}', \tilde{X}')$ and $f_2 : (G^{\text{ad}}, X^{\text{ad}}) = (G_1, X_1) \hookrightarrow (G'_2, X'_2)$ defined naturally by f . Let $\mathcal{N}_1^{\text{ad}}$ be the normalization of $\mathcal{N}'_{2O(G_1, X_1, p)}$ in $\text{Sh}_{H^{\text{ad}}}(G^{\text{ad}}, X^{\text{ad}})$. We know that $\mathcal{N}_1^{\text{ad}}$ is a normal integral model of $\text{Sh}_{H^{\text{ad}}}(G^{\text{ad}}, X^{\text{ad}}) = \text{Sh}_{H_1}(G_1, X_1)$ over $O(G_1, X_1, p)$ having the extension property, cf. 3.1.1 1). Moreover we have a commutative diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\tilde{s}'} & \tilde{\mathcal{N}}'_{O(G_1, X_1, p)} \\ q \downarrow & & \downarrow q_2 \\ \mathcal{N}_1^{\text{ad}} & \xrightarrow{s'_2} & \mathcal{N}'_{2O(G_1, X_1, p)}, \end{array}$$

where \tilde{s}' and s'_2 are naturally defined by \tilde{f}' and respectively by f'_2 and where q and q_2 are the natural morphisms whose existence is implied by the fact that \mathcal{N}'_2 has the extension property. We know that q_2 is a pro-étale cover of its image (cf. 6.2.1 and [Va4, 5.3.1]), that \tilde{s}' and s'_2 are finite morphisms (cf. 3.1.1 3)), that the generic fibres of \tilde{s}' and s'_2 are closed embeddings (cf. [Va1, 3.2.14]) and that the generic fibre of q is a pro-étale morphism (this last thing is well known). We easily get that q is a pro-étale cover of its image IM and that IM is an open closed subscheme of $\mathcal{N}_1^{\text{ad}}$. So IM is also a regular, formally smooth $O(G_1, X_1, p)$ -scheme. The connected components of $\mathcal{N}_1^{\text{ad}}$ are permuted transitively

by $G_1(\mathbb{A}_f^{(p)})$ (cf. [Va1, 3.3.2]) and so $\mathcal{N}_1^{\text{ad}}$ is the union of the $G_1(\mathbb{A}_f^{(p)})$ -translates of IM . So $\mathcal{N}_1^{\text{ad}}$ is a regular, formally smooth $O(G_1, X_1, p)$ -scheme and so an integral canonical model of $\text{Sh}_{H_1}(\mathcal{G}^{\text{ad}}, X^{\text{ad}}) = \text{Sh}_{H_1}(G_1, X_1)$ over $O(G_1, X_1, p)$. As $\mathcal{N}'_{2O(G_1, X_1, p)}$ is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme (cf. 6.2.1) and as s'_2 is a finite morphism, we get that $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, quasi-projective $O(G_1, X_1, p)$ -scheme. So 6.2 (a) holds.

If $(G_1, X_1) = (G_1^i, X_1^i)$ has compact factors, then so does (G'_2, X'_2) and so (cf. 6.2.1) $\tilde{\mathcal{N}}'_{O(G_1, X_1, p)}$ is a pro-étale cover of a smooth, projective $O(G_1, X_1, p)$ -scheme. It is easy to see that this implies that $\mathcal{N}_1^{\text{ad}}$ is a pro-étale cover of a smooth, projective $O(G_1, X_1, p)$ -scheme. So 6.2 (b) also holds. This ends the argument for 6.2 (a) and (b).

6.2.4. End of the proof of 1.6 2). We now come back to the general case of an arbitrary finite set I_0 and we will check the following extra thing:

(*) *the normalization of \mathcal{N}_1 of $\mathcal{N}_1^{\text{ad}}$ in $\text{Sh}_{H_1}(G_1, X_1)$ is the integral canonical model of $\text{Sh}_{H_1}(G_1, X_1)$ over $O(G_1, X_1, p)$.*

As in the proof of [Va4, 5.4.1], it suffices to check (*) under the extra assumption that $G_3 = G_1$. Based on 6.2 (v), to prove (*) we can also assume that I_0 has only one element i and so that (G_1, X_1) is a simple, adjoint Shimura pair. But in this case, (*) follows easily from 6.2 (vii) and the three Cases considered in 6.2.1 to 6.2.3.

From 6.2 (b) and from (*) we get that if (G_1^i, X_1^i) has compact factors for all $i \in I_0$, then \mathcal{N}_1 is a pro-étale cover of a smooth, projective $O(G_1, X_1, p)$ -scheme. This ends the proof of 1.6 2) and so it also ends the proof of 1.6.

6.3. New examples of Néron models. Let D be a DVR. Let K_D be the field of fractions of D . Let Z_{K_D} be a smooth, separated K_D -scheme of finite type. We recall (cf. [BLR, p. 12]) that a Néron model of Z_{K_D} over D is a smooth, separated D -scheme Z of finite type having Z_{K_D} as its generic fibre and which satisfies the following universal property, called the Néron mapping property:

(NMP) *for any smooth D -scheme Y and for every K_D -morphism $u_{K_D} : Y_{K_D} \rightarrow Z_{K_D}$, there is a unique morphism $u : Y \rightarrow Z$ of D -schemes extending u_{K_D} .*

A classical result of Néron says that any abelian variety over K_D has a Néron model over D , cf. [Né]. This result has an analogue for the case of K_D -torsors of smooth group schemes over K_D of finite type, cf. [BLR, Cor. 4 of p. 158]. On [BLR, p. 15] it is stated that the importance of the notion of Néron models “seems to be restricted” to “torsors under group schemes”. It was a deep insight of Milne which implicitly pointed out that Néron models are very important in the study of Shimura varieties, cf. the definitions [Mi2, 2.1, 2.2, 2.5 and 2.9]. The goal of this section is to bring to a concrete fruition Milne’s insights. In other words, we will use integral canonical models of certain Shimura varieties to provide large classes of projective varieties over certain K_D ’s which have projective Néron models and which often are not even embeddable into abelian varieties (or torsors of smooth group schemes) over K_D .

Let (G_1, X_1) be a Shimura pair of preabelian type such that each simple factor of $(G_1^{\text{ad}}, X_1^{\text{ad}})$ has compact factors. Let $p \in \mathbb{N}$ be a prime such that the group $G_{1\mathbb{Q}_p}$ is

unramified. Let H_1 be a hyperspecial subgroup of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$. If $p = 2$ we assume that the set $S(G_1, X_1, 2)$ defined in 1.6 2) is non-empty. Let $O(G_1, X_1, p)$ be as in 1.6 2). Let \mathcal{N}_1 be the integral canonical model of $\mathrm{Sh}_{H_1}(G_1, X_1)$ over $O(G_1, X_1, p)$, cf. 1.6 2). Let $H_{1,p}$ be a compact, open subgroup of $G_1(\mathbb{A}_f^{(p)})$ such that \mathcal{N}_1 is a pro-étale cover of $\mathcal{N}_1/H_{1,p}$. The $O(G_1, X_1, p)$ -scheme $\mathcal{N}_1/H_{1,p}$ is smooth and projective, cf. 1.6 2).

6.3.1. Theorem. *We take D to be an $O(G_1, X_1, p)$ -algebra which is a DVR of mixed characteristic $(0, p)$ and of index of ramification at most $\max\{1, p-2\}$. Let*

$$Z := \mathcal{N}_1/H_{1,p} \times_{O(G_1, X_1, p)} D.$$

Then the D -scheme Z is the Néron model of its generic fibre Z_{K_D} .

Proof: Let Y be a smooth D -scheme. Let $u_{K_D} : Y_{K_D} \rightarrow Z_{K_D}$ be a morphism of K_D -schemes. The D -scheme Z is projective and so there is an open subscheme U of Y such that the complement of U in Y has codimension at most 2 in Y and u_{K_D} extends uniquely to a morphism $v_U : U \rightarrow Z$. The scheme $U_1 := U \times_Z \mathcal{N}_{1D}$ is a pro-étale cover of U and so from the classical purity theorem (see [SGA1, p. 275]) we get that it extends to a pro-étale cover Y_1 of Y . The D -scheme Y_1 is healthy regular, cf. 1.2.1. So as \mathcal{N}_1 has the extension property, the natural morphism $Y_{1K_D} \rightarrow \mathcal{N}_1$ extends uniquely to a morphism $Y_1 \rightarrow \mathcal{N}_1$ of D -schemes. This implies that v_U extends uniquely to a morphism $u : Y \rightarrow Z$ of D -schemes. This ends the proof.

6.3.2. Remark. If $H_{1,p}$ is small enough, then $\mathrm{Sh}(G_1, X_1)_{\mathbb{C}}/H_{1,p} \times H_1$ is of general type (see [Mi1, 1.2 of §2]). So Z is not among the Néron models studied in [BLR]. Often the albanese variety of any connected component \mathcal{C}_1 of $\mathrm{Sh}(G_1, X_1)_{\mathbb{C}}/H_{1,p} \times H_1$ is trivial and so $Z_{K_D}^{\mathrm{an}}$ is not even embeddable into an abelian variety. For instance, this happens if $G_{1\mathbb{R}}^{\mathrm{ad}}$ is $SU^{\mathrm{ad}}(a, b)_{\mathbb{R}} \times_{\mathbb{R}} SU(a+b, 0)_{\mathbb{R}}^{\mathrm{ad}}$, with $a, b > 2$ (we have $H^{1,0}(\mathcal{C}_1, \mathbb{C}) = 0$, cf. [CL, p. 11–12]). This remark was first hinted at in [Va2].

6.4. Complements. 1) One can use [Va3] to show that the part of 6.1.2.2 pertaining to $p = 2$ and (G_1, X_1) a simple, adjoint Shimura pair of C_n type and having compact factors, can be entirely adapted for the $D_n^{\mathbb{H}}$ types.

2) Let the quadruple $(f, L, v, G_{\mathbb{Z}_{(p)}})$ be as in 1.3. We assume that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and that one of the conditions (i) and (ii) of 1.6 2) holds; so $p \geq 3$. Then from 4.2.2 2) and 1.6 1) we get that the triple (f, L, v) is a standard Hodge situation in the sense of [Va5, 1.4]. One can combine 4.2.2 2) with (iii) of 1.6 1) to get a weaker form of this for $p = 2$.

3) The arguments of 6.2 can be used to show starting from 4.5 that for $p \geq 3$ (resp. for $p = 2$) and for any triple (G_1, X_1, H_1) with (G_1, X_1) a Shimura pair of preabelian (resp. of abelian) type and with H_1 a hyperspecial subgroup of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$, $\mathrm{Sh}_{H_1}(G_1, X_1)$ has a “good” smooth integral model \mathcal{N}_1 over $E(G_1, X_1)_{(p)}$. For instance, if $G_1 = G_1^{\mathrm{ad}}$, then here by “good” we mean that a pro-étale cover of an open closed subscheme of \mathcal{N}_1 is a weak integral canonical model of $\mathrm{Sh}_H(G, X)$ over $E(G, X)_{(p)}$, where (G, X) is a Shimura pair of Hodge type having (G_1, X_1) as its adjoint and such that $G_{\mathbb{Q}_p}$ is unramified and where H is the unique hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ satisfying $H^{\mathrm{ad}} = H_1$.

Warning: the methods of [Va1] and of this paper can be used (in a way entirely similar to the proof of 6.3.1) to show that \mathcal{N}_1 is a weak integral canonical model only if \mathcal{N} is a pro-étale cover of a smooth, projective scheme over $E(G, X)_{(p)}$ (to be compared with the proof of [Va1, 6.2.2 a)] where the arguments did appeal to stronger extension properties than the smooth extension property, like the extension property). However, in part II we will show that always \mathcal{N}_1 is in fact an integral canonical model.

4) We consider a Shimura pair (G_1, X_1) of abelian type. We assume that $p = 2$, that all simple factors of $(G_1^{\text{ad}}, X_1^{\text{ad}})$ are of some $D_n^{\mathbb{H}}$ type and do not have compact factors and that $G_{1\mathbb{Q}_p}$ is unramified. Let H_1 be an arbitrary hyperspecial subgroup of $G_{1\mathbb{Q}_p}(\mathbb{Q}_p)$. Then entirely as in 6.2.2.2 we argue based on 3.1.4 (as a substitute of 6.1), that $\text{Sh}_{H_1}(G_1, X_1)$ has an integral canonical model over $E(G_1, X_1)_{(p)}$.

5) One can easily adapt 6.1 (resp. 3.3 and 6.1) to show that in the context of 1.5, if $p = 2$ (resp. if $p \geq 3$ and each simple factor of $(G^{\text{ad}}, X^{\text{ad}})$ has compact factors), then $\mathcal{N} = \mathcal{N}'$ is a pro-étale scheme over a smooth, projective $O_{(v)}$ -scheme. Also for $p \geq 3$ one can combine 1.5 with [Va1, Th. 5.1] to get variants of 1.5 in which in fact we have $\mathcal{N} = \mathcal{N}'$ without assuming that each simple factor of $(G^{\text{ad}}, X^{\text{ad}})$ has compact factors.

6) We now explain why we view 1.5 and its variants of 5) as a generalization of [Mo] for the unramified context with $p > 2$. We refer to 1.5 with $p > 2$. Let $G_{\mathbb{Z}_p}^1$ be as in 1.2 3). Let $G_{\mathbb{Q}_p}^2$ be the normal, semisimple subgroup of $G_{\mathbb{Q}_p}^{\text{der}}$ such that $G_{\mathbb{Q}_p}^{\text{ad}}$ is naturally isomorphic to $G_{\mathbb{Q}_p}^{\text{1ad}} \times_{\mathbb{Q}_p} G_{\mathbb{Q}_p}^{\text{2ad}}$. Let $g_1, \dots, g_n \in G_{\mathbb{Q}_p}^2(\mathbb{Q}_p)$. Let

$$\tilde{H} := H \cap g_1 H g_1^{-1} \cap \dots \cap g_n H g_n^{-1}.$$

Using Hodge quasi products as in [Va4, 2.4] and Segre embeddings as in [Va1, Example 3 of 2.5], one gets that there is an injective map $\tilde{f} : (G, X) \hookrightarrow (GSp(W, \psi), S)$ and a \mathbb{Z} -lattice \tilde{L} of W such that the following things hold:

– we have a perfect alternating form $\psi : \tilde{L} \otimes_{\mathbb{Z}} \tilde{L} \rightarrow \mathbb{Z}$, $\tilde{H} = \{h \in G(\mathbb{Q}_p) | h(\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p\}$, and $G_{\mathbb{Z}_p}^1$ is a closed subgroup of the Zariski closure of $G_{\mathbb{Q}_p}$ in $GL(\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

From 1.5 we get that $\text{Sh}_{\tilde{H}}(G, X)$ has a weak integral model $\tilde{\mathcal{N}}$ over $O_{(v)}$. As a scheme $\tilde{\mathcal{N}}$ is a quasi-finite scheme over a product \mathcal{N}^n of n -copies of the $O_{(v)}$ -scheme \mathcal{N} (cf. 3.1.1 1)) in such a way that the natural projection morphisms $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ are formally étale (this follows easily from 1.4 2) and 4.3 (iii)). If each simple factor of $(G^{\text{ad}}, X^{\text{ad}})$ has compact factors, then $\tilde{\mathcal{N}}$ is a pro-étale scheme over a smooth, projective $O_{(v)}$ -scheme (cf. 5)) and the natural projection morphisms $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ are étale covers.

Let $H_{\infty} := \bigcap_{g \in G_{\mathbb{Q}_p}^2} g H g^{-1}$. The projective limit of $\tilde{\mathcal{N}}$'s is the weak integral canonical model of $\text{Sh}_{H_{\infty}}(G, X)$ over $O_{(v)}$. If $(G_{\mathbb{Q}_p}/Z^0(G_{\mathbb{Q}_p})G_{\mathbb{Q}_p}^1)(\mathbb{Q}_p)$ has no compact subgroups normalized by $G_{\mathbb{Q}_p}^2(\mathbb{Q}_p)$, then $H_{\infty} = H \cap (Z^0(G_{\mathbb{Q}_p})G_{\mathbb{Q}_p}^1)(\mathbb{Q}_p)$.

One can also vary L subject to the fact that ψ induces a perfect form on it and $G_{\mathbb{Z}_p}^1$ is a reductive subgroup of $GL(L \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. So in the previous paragraph, often one can replace H_{∞} by many other compact subgroups of it containing $G_{\mathbb{Z}_p}^1(\mathbb{Z}_p)$. A quite similar approach was used in [Mo] for the particular case when $(G_1^{\text{ad}}, X_1^{\text{ad}})$ is a Shimura curve and

v_1^{ad} is a prime such that G_1^{ad} splits over $E(G_1^{\text{ad}}, X_1^{\text{ad}})_{v_1^{\text{ad}}}$. Warning: in our case v_1 is always unramified over an odd prime p while loc. cit. does not require this restriction.

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